Exam 1

Linear Algebra, Dave Bayer, February 13, 2007

Name: Answer Key

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation.

[1] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & -2 \end{array}\right] \begin{matrix} \\ \\ -2\boxed{1} \end{matrix}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & -2 \end{array}\right] \begin{matrix} \\ \\ -\boxed{2} \end{matrix}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \begin{matrix} \\ \\ \div(-4) \end{matrix}$$

$$\Downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \begin{matrix} -\boxed{3} \\ -2\boxed{3} \\ \\ \end{matrix}$$

check
$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} / 2 \stackrel{?}{=} \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} / 2 \quad ✓$$

answer: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1 \\ 1/2 \end{bmatrix}$

[2] What is the set of all solutions to each of the following systems of equations?

A $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ B

$$\left[\begin{array}{cccc|cc} 1 & 2 & 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 2 & 4 & 1 & 1 & 1 & 2 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|cc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{l} \\ -\boxed{1} \\ -2\boxed{1} \end{array}$$

$$\Downarrow$$

$$\left[\begin{array}{cccc|cc} 1 & 2 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right] \begin{array}{l} \\ \\ -\boxed{2} \end{array}$$

$\boxed{\text{A has \underline{no} solutions}}$

B

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

free cols

particular

homogeneous

$$\boxed{\text{B} \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

✓ ✓ ✓

[3] Use Gaussian elimination to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & -1 & -2 & 1 & -2 & 0 \end{array}\right] \begin{array}{l} \\ \\ -2\boxed{1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 0 & 1 & -1 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} -\boxed{2} \\ \\ +\boxed{2} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 6 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] \begin{array}{l} +2\boxed{3} \\ -3\boxed{3} \\ \\ \end{array}$$

check: $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -3 & 1 \\ -3 & 6 & -2 \\ 1 & -2 & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

$$\boxed{A^{-1} = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 6 & -2 \\ 1 & -2 & 1 \end{bmatrix}}$$

[4] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix}2 & 3\\ 1 & 3\end{bmatrix}$$

$$\begin{bmatrix}2&3\\1&3\end{bmatrix}\quad\begin{bmatrix}1&3\\2&3\end{bmatrix}\quad\begin{bmatrix}1&3\\0&-3\end{bmatrix}-2\boxed{1}\quad\begin{bmatrix}1&3\\0&1\end{bmatrix}*\tfrac{-1}{3}\quad\begin{bmatrix}1&0\\0&1\end{bmatrix}-3\boxed{2}$$

$$\boxed{A = \underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\2&1\end{bmatrix}}_{\begin{bmatrix}2&1\\1&0\end{bmatrix}}\underbrace{\begin{bmatrix}1&0\\0&-3\end{bmatrix}\begin{bmatrix}1&3\\0&1\end{bmatrix}}_{\begin{bmatrix}1&3\\0&-3\end{bmatrix}}}$$

check

$$\underbrace{\begin{bmatrix}2&1\\1&0\end{bmatrix}\begin{bmatrix}1&3\\0&-3\end{bmatrix}}_{\begin{bmatrix}2&3\\1&3\end{bmatrix}}\ ✓$$

[5] Express A^{-1} as a product of elementary matrices, where

$$A = \begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \Leftarrow \begin{bmatrix}1&\frac12&0\\0&1&0\\0&0&1\end{bmatrix} \Leftarrow \begin{bmatrix}1&\frac12&0\\0&1&\frac12\\0&0&1\end{bmatrix} \Leftarrow \begin{bmatrix}1&\frac12&0\\0&1&\frac12\\0&0&2\end{bmatrix} \Leftarrow \begin{bmatrix}1&\frac12&0\\0&2&1\\0&0&2\end{bmatrix} \Leftarrow \begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}$$

$$\boxed{A^{-1} = \begin{bmatrix}1&-\frac12&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&-\frac12\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&\frac12\end{bmatrix}\begin{bmatrix}1&0&0\\0&\frac12&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac12&0&0\\0&\uparrow&0\\0&0&1\end{bmatrix}}$$

$$\underbrace{\begin{bmatrix}1&-\frac12&\frac14\\0&1&-\frac12\\0&0&1\end{bmatrix} \qquad \begin{bmatrix}\frac12&&\\&\frac12&\\&&\frac12\end{bmatrix}}$$

check

$$\begin{bmatrix}2&1&0\\0&2&1\\0&0&2\end{bmatrix}\cdot\begin{bmatrix}\frac12&-\frac14&\frac18\\0&\frac12&-\frac14\\0&0&\frac12\end{bmatrix} \overset{?}{=} \begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$$