$\boxed{5}\boxed{5}\boxed{5}$ ← sorting code, from list as you hand in exam

Exam 1

Linear Algebra, Dave Bayer, February 15, 2005

Name: Solutions

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[1] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{c|c|c} \boxed{0} & \boxed{1} & \boxed{0} \\ ? & ? & ? \\ ? & ? & ? \\ \boxed{0} & \boxed{0} & \boxed{1} \end{array}\right]$$

starting pattern

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

← swap rows

$$\xrightarrow{③ = ③ - ②} \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{② = ② - ①} \left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{① = ① - ②} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

free cols

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ 1 & 0 \\ 0 & \\ 0 & \end{array}\right]$$

particular homogeneous

Solution

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ -1 \\ +1 \end{bmatrix}$$

check:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} s \\ 1 \\ 1-t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 3 \end{bmatrix} \quad \checkmark$$

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[2] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -2 & 2 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -2 & 2 & -3 \end{bmatrix} \xrightarrow{1 \rightleftarrows 2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ -2 & 2 & -3 \end{bmatrix} \xrightarrow{3 = 3 + 2(1)} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{3 = 3 - 2(2)} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{3 = -3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{1 = 1 - 3(3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{2 = 2 - 2(3)} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$1 \rightleftarrows 2$, $3 = 3 - 2(1)$, $3 = 3 + 2(2)$, $3 = -3$, $1 = 1 + 3(3)$, $2 = 2 + 2(3)$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -2 & 2 & -3 \end{bmatrix}$$

answer

check:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -2 & 2 & -3 \end{bmatrix} \checkmark$$

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[3] What is the determinant of the following 4×4 matrix?

answer $\Rightarrow$ $$\boxed{\det \begin{bmatrix} 1 & 2 & a & c \\ 2 & 1 & b & d \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} = 21}$$

block upper triangular, so det is product of block dets,

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = (-3)(-7) = 21$$

---

check:

$$\begin{vmatrix} 1 & 2 & a & c \\ 2 & 1 & b & d \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{vmatrix} = +1 \begin{vmatrix} 1 & b & d \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & a & c \\ 0 & 3 & 4 \\ 0 & 4 & 3 \end{vmatrix}$$

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

$$= 1 \cdot 1 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} - 2 \cdot 2 \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix}$$

$$= \underbrace{(1 \cdot 1 - 2 \cdot 2)}_{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = (-3)(-7) = 21$$

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[**4**] Using Cramer's rule, find w satisfying the following system of equations:

$$\begin{bmatrix} 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$w = \frac{\begin{vmatrix} 2 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 0 & -1 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 \end{vmatrix}}$$

$$= 2\begin{vmatrix} -1 & 3 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & -1 \end{vmatrix} - 0\left|\Sigma\right|$$

$$= 1\begin{vmatrix} -1 & 3 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{vmatrix} - \begin{vmatrix} 3 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & -1 & -1 \end{vmatrix} + \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & -1 \end{vmatrix} - \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

expand 1st cols

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

① = ① + ③
② = ② − ③

$$\begin{vmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ -1 & -1 & -1 \end{vmatrix}$$

$+(-16)$

col swap $-(16)$ col swap $+(-16)$

$+$ -9 -1 -1
$-$ 3 3 -1

$+$ 27 -1 -1
$-$ 3 3 3

$-(16)$

$$w = \frac{2(-16) - (-1)(16) + (-1)(-16) - 0(16)}{1(-16) - 1(16) + (-16) - (16)} = \frac{0}{-64} = 0 \qquad \boxed{w = 0}$$

answer

check $w = 0 \Rightarrow (x, y, z) = (\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4})$

$$\left[\begin{array}{cccc|c} 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 4 & -1 \\ 1 & -1 & -1 & -1 & 0 \end{array}\right]$$

①, ②, ③ = " − ④

$$\begin{bmatrix} 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \\ -1/4 \\ -1/4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} \checkmark$$

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(try again)

[4] Using Cramer's rule, find w satisfying the following system of equations:

$$\begin{bmatrix} 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$w = \frac{\begin{vmatrix} 2 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \\ 0 & -1 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \\ 1 & -1 & -1 & -1 \end{vmatrix}} \xRightarrow[\substack{② = ② - ④ \\ ③ = ③ - ④}]{① = ① - ④} \frac{\begin{vmatrix} 2 & 4 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & -1 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 1 & -1 & -1 & -1 \end{vmatrix}}$$

Numerator: sum to 0 4 4 4 so singular, det = 0

Denominator: (= −64)

$\boxed{w = 0}$

check:

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & -1 & 2 \\ 1 & -1 & 3 & -1 & -1 \\ 1 & -1 & -1 & 3 & -1 \\ 1 & -1 & -1 & -1 & 0 \end{array}\right]$$

$\Downarrow$ ① = ① − ④, ② = ② − ④, ③ = ③ − ④

$$\left[\begin{array}{cccc|c} 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 4 & -1 \\ 1 & -1 & -1 & -1 & 0 \end{array}\right]$$

$\Downarrow$ divide out 4's, rearrange rows

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & -1/4 \end{array}\right] \xRightarrow[\text{①} = \text{①} + \text{②} + \text{③} + \text{④}]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & -1/4 \end{array}\right]$$

w=0 ✓

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[**5**] Give a formula for the matrix which is inverse to:

answer $\Rightarrow$ $$\begin{bmatrix} 1 & a & 0 & -1 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & +ab & 1-abc \\ 0 & 1 & -b & +bc \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & a & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & b & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$[A|I]$

$\downarrow$

$\vdots$

$\downarrow$

$[I|A^{-1}]$

$1 = 1 - a2$

$2 = 2 - b3$

$3 = 3 - c4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -ab & -1 & 1 & -a & 0 & 0 \\ 0 & 1 & 0 & -bc & 0 & 1 & -b & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$1 = 1 + ab3$

$2 = 2 + bc4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & -a & +ab & -abc \\ 0 & 1 & 0 & 0 & 0 & 1 & -b & +bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$1 = 1 + 4$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -a & +ab & 1-abc \\ 0 & 1 & 0 & 0 & 0 & 1 & -b & +bc \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

check!

$$\begin{bmatrix} 1 & a & 0 & -1 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & +ab & 1-abc \\ 0 & 1 & -b & +bc \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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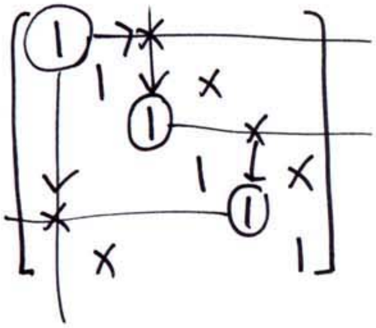
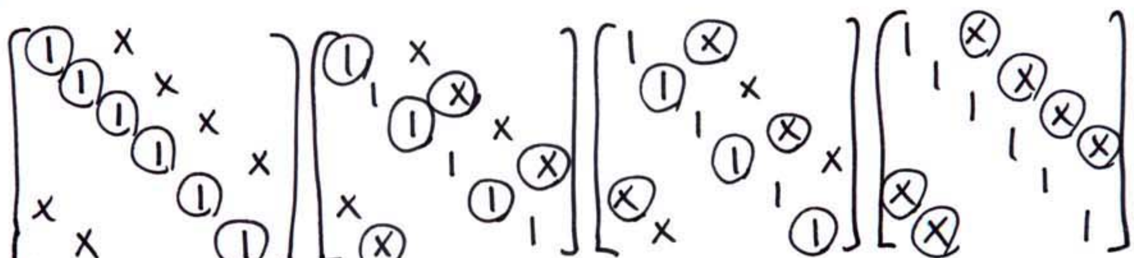
[**6**] What is the determinant of the following 6×6 matrix? What is the determinant of the corresponding $n \times n$ matrix?



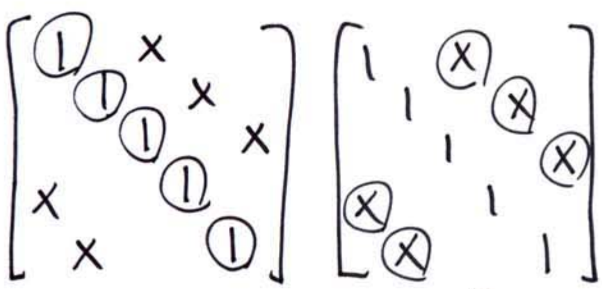
$$\begin{bmatrix} 1 & 0 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 \\ 0 & 0 & 0 & 1 & 0 & x \\ x & 0 & 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 & 0 & 1 \end{bmatrix}$$

n even: 4 terms to determinant



1 (no swaps) $\quad + x^{n/2}$ (6 swaps) $\quad + x^{n/2}$ (6 swaps) $\quad + x^n$ (8 swaps)

n odd: 2 terms to determinant



$1 \quad + x^n$ (6 swaps)

$$\boxed{\text{so } \det = \begin{cases} (1+x^{n/2})^2, & n \text{ even} \\ 1+x^n, & n \text{ odd} \end{cases} \qquad \det = (1+x^3)^2 \text{ above}}$$

check: $x=-1 \Rightarrow \underbrace{\text{rows sum to } 0}_{\text{singular}}$, and $(1+(-1)^3)^2 = 0$ ✓

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```
a = {{1, 0, x, 0, 0},
   {0, 1, 0, x, 0},
   {0, 0, 1, 0, x},
   {x, 0, 0, 1, 0},
   {0, x, 0, 0, 1}};
a // MatrixForm
Det[a]
```

$$\begin{pmatrix} 1 & 0 & x & 0 & 0 \\ 0 & 1 & 0 & x & 0 \\ 0 & 0 & 1 & 0 & x \\ x & 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 & 1 \end{pmatrix}$$

$1 + x^5$

```
a = {{1, 0, x, 0, 0, 0},
   {0, 1, 0, x, 0, 0},
   {0, 0, 1, 0, x, 0},
   {0, 0, 0, 1, 0, x},
   {x, 0, 0, 0, 1, 0},
   {0, x, 0, 0, 0, 1}};
a // MatrixForm
Det[a]
```

$$\begin{pmatrix} 1 & 0 & x & 0 & 0 & 0 \\ 0 & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & x & 0 \\ 0 & 0 & 0 & 1 & 0 & x \\ x & 0 & 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 & 0 & 1 \end{pmatrix}$$

$1 + 2\,x^3 + x^6$