# Final Exam

Linear Algebra, Dave Bayer, December 22, 2022

**Name:** ______________________________ **Uni:** ______________

| [1] | [2] | [3] | [4] | [5] | [6] | [7] | [8] | **Total** |
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If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} t$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

check:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \left( \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} t \right)$$

$$\downarrow \qquad \downarrow$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} t$$

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There are many possible answers.
A full rank answer that checks is correct.

[2] Find the $3 \times 3$ matrix $A$ that projects $\mathbb{R}^3$ orthogonally onto the hyperplane $x + y + 3z = 0$.

(Taken from 8:40 Exam 2 [4])

$$A = \frac{1}{11}\begin{bmatrix} 10 & -1 & -3 \\ -1 & 10 & -3 \\ -3 & -3 & 2 \end{bmatrix}$$

$x + y + 3z = 0$

$(1,1,3)\cdot(x,y,z) = 0$

$A + B = I$, $B$ projects onto normal vector $(1,1,3)$

$$(x,y,z) \overset{B}{\longmapsto} \frac{(x,y,z)\cdot(1,1,3)}{(1,1,3)\cdot(1,1,3)}(1,1,3)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{B}{\longmapsto} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \end{bmatrix}/11 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \end{bmatrix}/11 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{bmatrix}/11$$

$$A = I - B = \begin{bmatrix} 11 & & \\ & 11 & \\ & & 11 \end{bmatrix}/11 - \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{bmatrix}/11 = \begin{bmatrix} 10 & -1 & -3 \\ -1 & 10 & -3 \\ -3 & -3 & 2 \end{bmatrix}/11$$

check:

$$\begin{bmatrix} 10 & -1 & -3 \\ -1 & 10 & -3 \\ -3 & -3 & 2 \end{bmatrix}/11 \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 11 & 33 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix}/11 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

in plane

⊥ to plane

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[3] Find the $3 \times 3$ matrix $A$ that projects $\mathbb{R}^3$ orthogonally onto the plane $x + y + z = 0$, with respect to the inner product

$$< (a, b, c),\ (r, s, t) > \quad = \quad \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

(Taken from 10:10 Exam 2 [5])

$$A = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ 1 & 4 & 1 \\ -2 & -2 & 1 \end{bmatrix}$$

$(1,1,1)$ is normal to $x+y+z=0$ using the usual dot product.
It is **not** normal to the plane using this inner product.
We'll still use $A = I - B$ but we need to first find the normal.

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2,-1,2)$$

(two vectors in plane)

looks familiar for a reason

$$\langle (x,y,z), (2,-1,2) \rangle = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\langle (2,-1,2), (2,-1,2) \rangle = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 3$$

$$(x,y,z) \overset{B}{\longmapsto} \frac{\langle (x,y,z), (2,-1,2) \rangle}{\langle (2,-1,2), (2,-1,2) \rangle} (2,-1,2)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{B}{\longmapsto} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \Big/ 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{so} \quad B = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 3$$

$$A = I - B = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix} \Big/ 3 - \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 1 & -2 & -2 \\ 1 & 4 & 1 \\ -2 & -2 & 1 \end{bmatrix} \Big/ 3$$

check:

$$\begin{bmatrix} 1 & -2 & -2 \\ 1 & 4 & 1 \\ -2 & -2 & 1 \end{bmatrix} \Big/ 3 \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 0 & 1 & \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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in plane

⊥ to plane for this inner product

[4] Find $A^n$ where $A$ is the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}$$

$$A^n = \frac{(-1)^n}{1} \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} + \frac{0^n}{1} \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\lambda = -1, 0 \qquad A^n = (-1)^n \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} + 0^n \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

[5] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$y = \frac{e^{t}}{1} \begin{bmatrix} -5 \\ 5 \end{bmatrix} + \frac{e^{2t}}{1} \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\lambda = 1, 2 \qquad e^{At} = e^{t} \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \qquad y = e^{t} \begin{bmatrix} -5 \\ 5 \end{bmatrix} + e^{2t} \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

[6] Find $e^{At}$ where $A$ is the matrix

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$e^{At} = \frac{1}{6}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 2 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^{3t}}{3}\begin{bmatrix} 0 & 4 & 4 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\lambda = 0, 2, 3 \qquad e^{At} = \frac{1}{6}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 4 & -2 \\ 0 & -4 & 2 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 2 & -2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^{3t}}{3}\begin{bmatrix} 0 & 4 & 4 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

[7] Find $e^{At}$ where $A$ is the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{At} = \frac{e^{3t}}{4} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \frac{e^{t}}{4} \begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix} + \frac{te^{t}}{2} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\lambda = 3, 1, 1 \qquad e^{At} = \frac{e^{3t}}{4} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} + \frac{e^{t}}{4} \begin{bmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{bmatrix} + \frac{te^{t}}{2} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

[8] Express the quadratic form

$$-2x^2 - 2xy - 3y^2 - 2xz - 3z^2$$

as a sum of squares of orthogonal linear forms.

$$-\tfrac{4}{3}\left(x+y+z\right)^2 + -\tfrac{3}{2}\left(y-z\right)^2 + -\tfrac{1}{6}\left(2x-y-z\right)^2$$

$\lambda = -4, -3, -1$ $\quad A = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -3 & 0 \\ -1 & 0 & -3 \end{bmatrix} = -\frac{4}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

$$-\frac{4}{3}(x+y+z)^2 - \frac{3}{2}(y-z)^2 - \frac{1}{6}(2x-y-z)^2$$