Final Exam

Linear Algebra, Dave Bayer, December 16, 2022

Name: ______________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} t$$

$$\boxed{\begin{bmatrix} 2 & -3 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 2 & -3 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} t \right)$$

$$\downarrow \qquad \downarrow$$

$$\begin{bmatrix} -3 \\ -6 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} t$$

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There are many possible answers.
A full rank answer that checks is correct.

[2] Find the 3×3 matrix A that projects $\mathbb{R}^3$ orthogonally onto the plane $x + y + 2z = 0$.

(Taken from 10:10 Exam 2 [4])

$$A = \frac{1}{6}\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

$$\underbrace{x+y+2z=0}_{(1,1,2)\cdot(x,y,z)=0}$$

$A+B=I$, B projects onto normal vector $(1,1,2)$

$$(x,y,z) \xmapsto{B} \frac{(x,y,z)\cdot(1,1,2)}{(1,1,2)\cdot(1,1,2)}(1,1,2)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{B} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \Big/ 6 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \Big/ 6 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \Big/ 6$$

$$A = I - B = \begin{bmatrix} 6 & & \\ & 6 & \\ & & 6 \end{bmatrix} \Big/ 6 - \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \Big/ 6 = \boxed{\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} \Big/ 6}$$

check:

$$\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix} \Big/ 6 \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 12 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix} \Big/ 6 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

in plane

⊥ to plane

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[3] Find the 3×3 matrix A that projects $\mathbb{R}^3$ orthogonally onto the plane $x + y + z = 0$, with respect to the inner product

$$< (a,b,c),\ (r,s,t) > \quad = \quad \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

(Taken from 8:40 Exam 2 [5])

$$A = \frac{1}{2} \begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

$(1,1,1)$ is normal to $x+y+z=0$ using the usual dot product.
It is **not** normal to the plane using this inner product.
We'll still use $A = I - B$ but we need to first find the normal.

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{\text{two vectors in plane}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2,-1,1)$$

$$\langle (x,y,z),(2,-1,1) \rangle = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}^{\text{looks familiar for a reason}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\langle (2,-1,1),(2,-1,1) \rangle = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2$$

$$(x,y,z) \overset{B}{\longmapsto} \frac{\langle (x,y,z),(2,-1,1) \rangle}{\langle (2,-1,1),(2,-1,1) \rangle} (2,-1,1)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{B}{\longmapsto} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \Big/ 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{so} \quad B = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 2$$

$$A = I - B = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \Big/ 2 - \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 2 = \boxed{\begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \Big/ 2}$$

check:

$$\begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \Big/ 2 \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Big/ 2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

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in plane

⊥ to plane for this inner product

[4] Find A^n where A is the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{A^n = \frac{0^n}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \frac{3^n}{3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}}$$

$$\lambda = 0, 3 \qquad A^n = \frac{0^n}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \frac{3^n}{3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$$

[5] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{y = \frac{e^{-2t}}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{e^{t}}{3} \begin{bmatrix} 2 \\ 4 \end{bmatrix}}$$

$$\lambda = -2, 1 \qquad e^{At} = \frac{e^{-2t}}{3} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} + \frac{e^{t}}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad y = \frac{e^{-2t}}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{e^{t}}{3} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

[6] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$e^{At} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} + \frac{e^{2t}}{1}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^{3t}}{3}\begin{bmatrix} 0 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\lambda = 0, 2, 3 \qquad e^{At} = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^{3t}}{3}\begin{bmatrix} 0 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

[7] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$e^{At} = \frac{1}{4}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} + \frac{e^{2t}}{4}\begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} + \frac{te^{2t}}{2}\begin{bmatrix} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = 0, 2, 2 \qquad e^{At} = \frac{1}{4}\begin{bmatrix} 0 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} + \frac{e^{2t}}{4}\begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} + \frac{te^{2t}}{2}\begin{bmatrix} 0 & 3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

[8] Express the quadratic form

$$3x^2 - 2xy + 2y^2 - 2xz + 2z^2$$

as a linear combination of squares of orthogonal linear forms.

$$\boxed{\;\boxed{1/3}\left(\boxed{x+y+z}\right)^2 + \boxed{1}\left(\boxed{y-z}\right)^2 + \boxed{2/3}\left(\boxed{2x-y-z}\right)^2\;}$$

$$\lambda = 1, 2, 4 \qquad A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{3}(x + y + z)^2 + (y - z)^2 + \frac{2}{3}(2x - y - z)^2$$