Exam 2, 8:40am

Linear Algebra, Dave Bayer, November 10, 2022

Name: ____ **Uni:** __________________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}\Big/20 \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

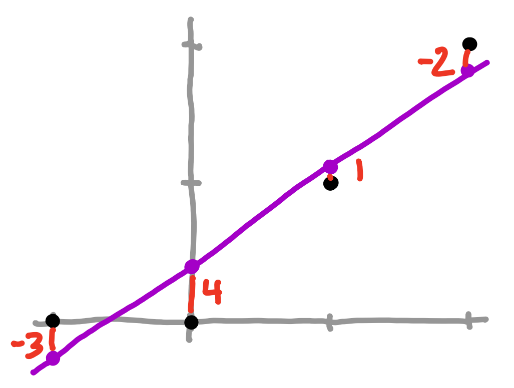
$$= \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\Big/10 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}\Big/10$$

$$\boxed{y = \tfrac{7}{10}x + \tfrac{2}{5}}$$

x^0	x^1	y	$ax+b$	Δ
1	-1	0	-3	-3
1	0	0	4	4
1	1	1	11	1
1	2	2	18	-2
			/10	/10

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} -3 \\ 4 \\ 1 \\ -2 \end{bmatrix} \checkmark \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -3 \\ 4 \\ 1 \\ -2 \end{bmatrix} \checkmark$$

need error to be $\perp$ to columns of matrix A



[2] Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 6 & 1 & 1 \\ 1 & 6 & 2 & 2 \\ 0 & 6 & 0 & 0 \\ 0 & 6 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{vmatrix} 1 & 6 & 1 & 1 \\ 1 & 6 & 2 & 2 \\ 0 & 6 & 0 & 0 \\ 0 & 6 & 2 & 5 \end{vmatrix} = -6\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{vmatrix} = -6\left(+1\begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}\right) = -6(1\cdot 6 - 1\cdot 3) = \boxed{-18}$$

$$\begin{vmatrix} 1 & 6 & 1 & 1 \\ 1 & 6 & 2 & 2 \\ 0 & 6 & 0 & 0 \\ 0 & 6 & 2 & 5 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{vmatrix}$$

check another way:

$$\begin{vmatrix} 1 & 6 & 1 & 1 \\ 1 & 6 & 2 & 2 \\ 0 & 6 & 0 & 0 \\ 0 & 6 & 2 & 5 \end{vmatrix} = 6\underbrace{\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 5 \end{vmatrix}}_{\text{pull 6 out from col 2}} = 6\underbrace{\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}}_{\text{subtract col 3 from col 4}} = 6\underbrace{\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix}}_{\text{subtract row 1 from row 2}} = 6\underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix}}_{\text{clean up cols 2 and 3}} = -18$$

☑

[3] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \qquad \underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} 6 & -3 & 0 \\ -5 & 5 & -1 \\ 2 & -2 & 1 \end{bmatrix}} \underset{\checkmark\checkmark}{/3} = \underset{I}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$$

$$\begin{array}{c|cccc|} 1 & 1 & 0 & 1 & 1 \\ \hline 1 & 2 & 2 & 1 & 2 \\ 1 & 2 & 5 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ \hline \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} 6 & -3 & 0 \\ -5 & 5 & -1 \\ 2 & -2 & 1 \end{bmatrix} /3}$$

[4] Find the 3×3 matrix A that projects R^3 orthogonally onto the hyperplane $\underbrace{x + y + 3z = 0}_{(1,1,3)\cdot(x,y,z)=0}$, with respect to the usual inner product.

$A + B = I$, B projects onto normal vector $(1,1,3)$

$$(x,y,z) \xmapsto{B} \frac{(x,y,z)\cdot(1,1,3)}{(1,1,3)\cdot(1,1,3)}(1,1,3)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{B} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \end{bmatrix} / 11 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \end{bmatrix} / 11 = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{bmatrix} / 11$$

$$A = I - B = \begin{bmatrix} 11 & & \\ & 11 & \\ & & 11 \end{bmatrix} / 11 - \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{bmatrix} / 11 = \boxed{\begin{bmatrix} 10 & -1 & -3 \\ -1 & 10 & -3 \\ -3 & -3 & 2 \end{bmatrix} / 11}$$

check:

$$\begin{bmatrix} 10 & -1 & -3 \\ -1 & 10 & -3 \\ -3 & -3 & 2 \end{bmatrix} / 11 \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 11 & 33 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{bmatrix} / 11 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

in plane

$\perp$ to plane

☑ ☑ ☑

[5] Find the 3×3 matrix A that projects R^3 orthogonally onto the hyperplane $x + y + z = 0$, with respect to the inner product

$$< (a,b,c),\ (r,s,t) > \quad = \quad \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

(1,1,1) is normal to $x+y+z=0$ using the usual dot product.
It is **not** normal to the plane using this inner product.
We'll still use $A = I - B$ but we need to first find the normal.

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{\text{two vectors in plane}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2,-1,1)$$

$$\langle (x,y,z),(2,-1,1) \rangle = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}^{\text{looks familiar for a reason}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\langle (2,-1,1),(2,-1,1) \rangle = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = 2$$

$$(x,y,z) \xmapsto{B} \frac{\langle (x,y,z),(2,-1,1) \rangle}{\langle (2,-1,1),(2,-1,1) \rangle} (2,-1,1)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{B} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \Big/ 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{so} \qquad B = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 2$$

$$A = I - B = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \Big/ 2 - \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix} \Big/ 2 = \boxed{\begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \Big/ 2}$$

check:

$$\begin{bmatrix} 0 & -2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix} \Big/ 2 \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \Big/ 2 = \begin{bmatrix} 0 & 1 & \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

☑ ☑ ☑

in plane

⊥ to plane for this inner product