Exam 2, 10:10am

Linear Algebra, Dave Bayer, November 10, 2022

Name: ____________________ **Uni:** ____________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix}-1&1\\0&1\\1&1\\2&1\end{bmatrix}}_{A}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}0\\1\\1\\2\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}-1&0&1&2\\1&1&1&1\end{bmatrix}\begin{bmatrix}-1&1\\0&1\\1&1\\2&1\end{bmatrix}}\begin{bmatrix}a\\b\end{bmatrix}=\underbrace{\begin{bmatrix}-1&0&1&2\\1&1&1&1\end{bmatrix}\begin{bmatrix}0\\1\\1\\2\end{bmatrix}}$$

$$\begin{bmatrix}6&2\\2&4\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}5\\4\end{bmatrix}$$

$$\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}6&2\\2&4\end{bmatrix}^{-1}\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}4&-2\\-2&6\end{bmatrix}\Big/20\begin{bmatrix}5\\4\end{bmatrix}$$

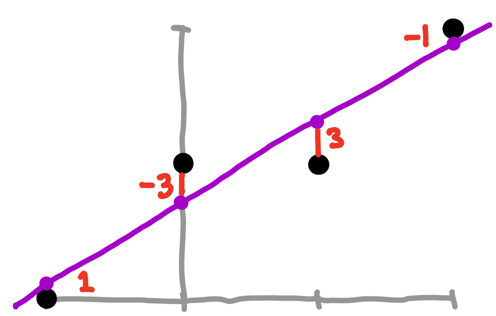
$$=\begin{bmatrix}2&-1\\-1&3\end{bmatrix}\Big/10\begin{bmatrix}5\\4\end{bmatrix}=\begin{bmatrix}6\\7\end{bmatrix}\Big/10$$

$$\boxed{y = \tfrac{3}{5}x + \tfrac{7}{10}}$$

x^0	x^1	y	$ax+b$	Δ
1	-1	0	1	1
1	0	1	7	-3
1	1	1	13	3
1	2	2	19 /10	-1 /10

$$\underbrace{\begin{matrix}1\\1\\1\\1\end{matrix}\ \perp\ \begin{matrix}1\\-3\\3\\-1\end{matrix}\ \boxed{\checkmark} \qquad \begin{matrix}-1\\0\\1\\2\end{matrix}\ \perp\ \begin{matrix}1\\-3\\3\\-1\end{matrix}\ \boxed{\checkmark}}$$

need error to be $\perp$ to columns of matrix A



[2] Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix} \qquad \begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 3 & 4 \end{vmatrix} = -5 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} = -5\left(1\begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}\right) = -5(1 \cdot 1 - 1(-1))$$

$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 3 & 4 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} \qquad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{vmatrix} \qquad = \boxed{-10}$$

check another way:

$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 5 & 5 & 5 & 5 \\ 0 & 0 & 3 & 4 \end{vmatrix} = 5 \underbrace{\begin{vmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 4 \end{vmatrix}}_{\text{pull 5 out of row 3}} = 5 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{vmatrix} = 5 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{vmatrix}$$

subtract col 3 from col 4

subtract col 1 from col 3

$$5 \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{vmatrix} = -5 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & 1 \end{vmatrix} = +5 \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -10 \quad ☑$$

[3] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{bmatrix} \qquad \underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 3 & 4 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} 1 & 1 & -1 \\ -4 & 4 & 2 \\ 3 & -3 & -1 \end{bmatrix}} \big/ 2 = \underset{I}{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}$$

$$\begin{array}{c|cccc|} 1 & 1 & 0 & 1 & 1 \\ \hline 2 & 1 & 3 & 2 & 1 \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 & 1 \\ \hline \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -4 & 4 & 2 \\ 3 & -3 & -1 \end{bmatrix} \big/ 2}$$

[4] Find the 3×3 matrix A that projects R^3 orthogonally onto the hyperplane $x + y + 2z = 0$, with respect to the usual inner product.

$$\underbrace{x+y+2z = 0}_{(1,1,2)\cdot(x,y,z)=0}$$

$A + B = I$, B projects onto normal vector $(1,1,2)$

$$(x,y,z) \xmapsto{B} \frac{(x,y,z)\cdot(1,1,2)}{(1,1,2)\cdot(1,1,2)}(1,1,2)$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} \xmapsto{B} \begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}1&1&2\end{bmatrix}/6\begin{bmatrix}x\\y\\z\end{bmatrix} \qquad B = \begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}1&1&2\end{bmatrix}/6 = \begin{bmatrix}1&1&2\\1&1&2\\2&2&4\end{bmatrix}/6$$

$$A = I - B = \begin{bmatrix}6&&\\&6&\\&&6\end{bmatrix}/6 - \begin{bmatrix}1&1&2\\1&1&2\\2&2&4\end{bmatrix}/6 = \boxed{\begin{bmatrix}5&-1&-2\\-1&5&-2\\-2&-2&2\end{bmatrix}/6}$$

check:

$$\begin{bmatrix}5&-1&-2\\-1&5&-2\\-2&-2&2\end{bmatrix}/6\begin{bmatrix}1&1&2\\1&-1&0\\2&0&-1\end{bmatrix} = \begin{bmatrix}0&6&12\\0&-6&0\\0&0&-6\end{bmatrix}/6 = \begin{bmatrix}0&1&2\\0&-1&0\\0&0&-1\end{bmatrix}$$

in plane

$\perp$ to plane

☑ ☑ ☑

[5] Find the 3×3 matrix A that projects $\mathbb{R}^3$ orthogonally onto the hyperplane $x + y + z = 0$, with respect to the inner product

$$< (a,b,c),\ (r,s,t) > \quad = \quad \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$(1,1,1)$ is normal to $x+y+z=0$ using the usual dot product.
It is **not** normal to the plane using this inner product.
We'll still use $A = I - B$ but we need to first find the normal.

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{\text{two vectors in plane}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (2,-1,2)$$

$$\langle (x,y,z),(2,-1,2) \rangle = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}^{\text{looks familiar for a reason}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\langle (2,-1,2),(2,-1,2) \rangle = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 3$$

$$(x,y,z) \xmapsto{B} \frac{\langle (x,y,z),(2,-1,2) \rangle}{\langle (2,-1,2),(2,-1,2) \rangle} (2,-1,2)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{B} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \Big/ 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{so} \quad B = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 3$$

$$A = I - B = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix} \Big/ 3 - \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 3 = \boxed{\begin{bmatrix} 1 & -2 & -2 \\ 1 & 4 & 1 \\ -2 & -2 & 1 \end{bmatrix} \Big/ 3}$$

check:

$$\begin{bmatrix} 1 & -2 & -2 \\ 1 & 4 & 1 \\ -2 & -2 & 1 \end{bmatrix} \Big/ 3 \begin{bmatrix} 2 & 1 & 1 \\ -1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

☑ ☑ ☑

in plane

$\perp$ to plane for this inner product