Exam 1, 10:10am

Linear Algebra, Dave Bayer, October 4, 2022

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Solve the following system of equations.

$$\begin{bmatrix}1&2&1&1\\1&1&2&1\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

2 independent conditions on 4 variables

$\Rightarrow$ solution is 2-dimensional (a parametrized plane)

$$\begin{bmatrix}1&2&1&1\\1&1&2&1\end{bmatrix}\left(\begin{bmatrix}0\\0\\3\\-3\end{bmatrix} + \overbrace{\begin{bmatrix}1&0\\0&1\\0&1\\-1&-3\end{bmatrix}}^{\to\, 0}\begin{bmatrix}s\\t\end{bmatrix}\right) = \begin{bmatrix}0\\3\end{bmatrix}$$

$$\boxed{\begin{bmatrix}w\\x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\3\\-3\end{bmatrix} + \begin{bmatrix}1&0\\0&1\\0&1\\-1&-3\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}}$$

[2] Using matrix multiplication, count the number of paths of length nine from x to itself.

$$\begin{array}{c|c} & \text{from} \\ & x \quad y \quad z \\ \begin{array}{c} x \\ y \\ z \\ \text{to} \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\underset{A}{\begin{bmatrix} \boxed{0} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} = \underset{A^2}{\begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}$$

1 path length 2 x to x

$$\underset{A}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{A^2}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}} = \underset{A^3}{\begin{bmatrix} \boxed{0} & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}}$$

0 paths length 3 x to x

$$\underset{A^3}{\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}} \underset{A^3}{\begin{bmatrix} 0 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}} = \underset{A^6}{\begin{bmatrix} \boxed{5} & & \\ 5 & & \\ 9 & & \end{bmatrix}}$$

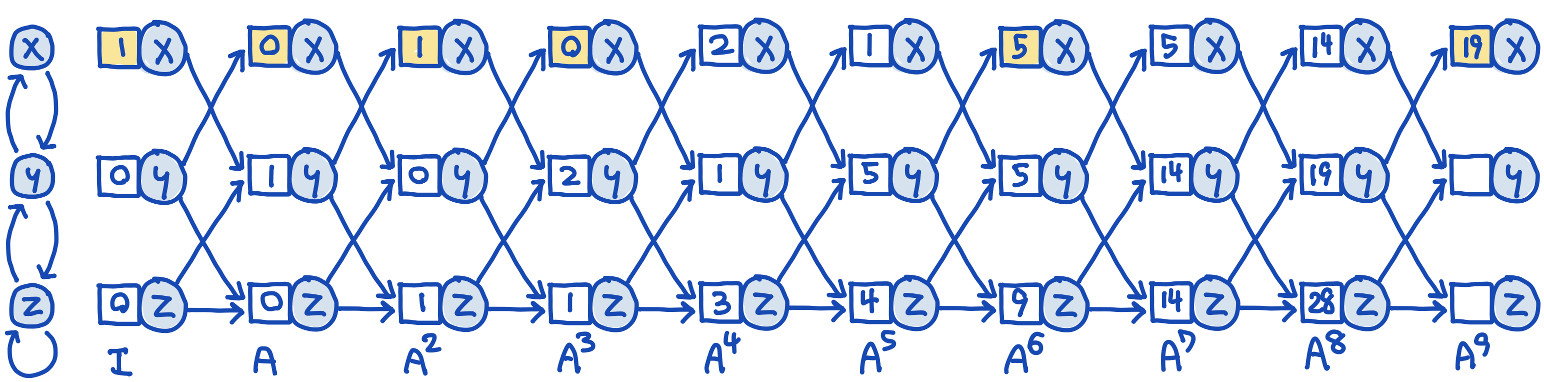
5 paths length 6 x to x

$$\underset{A^3}{\begin{bmatrix} \boxed{0 \quad 2 \quad 1} \\ 2 \quad 1 \quad 3 \\ 1 \quad 3 \quad 3 \end{bmatrix}} \underset{A^6}{\begin{bmatrix} \boxed{\begin{matrix} 5 \\ 5 \\ 9 \end{matrix}} & & \end{bmatrix}} = \underset{A^9}{\begin{bmatrix} \boxed{19} & & \\ & & \\ & & \end{bmatrix}}$$

19 paths length 9 x to x

$$\boxed{\text{19 paths of length 9}}$$

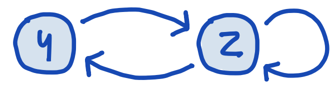
check:



We can unfold the graph, and count all paths by hand.
Start with one path of length zero "at" x.
Each count is the sum of the counts coming in.

With more time, one can understand this count better:

Call a path ⓧ to ⓧ "elementary" if it never visits ⓧ until it is done.
Every path ⓧ to ⓧ can be factored into elementary paths.
An elementary path steps to ⓨ, then is a path ⓨ to ⓨ that doesn't visit ⓧ, then returns to ⓧ.

So we need to count paths ⓨ to ⓨ for this subgraph: 

There is one elementary path ⓨ to ⓨ of each length 2, 3, 4, 5, 6, ...
To count all paths ⓨ to ⓨ we count all combinations of elementary paths.

length	paths	
0	1	
1	0	
2	1	2
3	1	3
4	2	22, 4
5	3	23, 32, 5
6	5	222, 24, 33, 42, 6
7	8	223, 232, 25, 322, 34, 43, 52, 7

$$
\begin{array}{rcccccc}
 & +1 & 1 & 1 & 1 & 1 & 1 \\
 & & +0 & 0 & 0 & 0 & 0 \\
\leftarrow 1 & & & +1 & 1 & 1 & 1 \\
 & \leftarrow 1 & & & +1 & 1 & 1 \\
 & & \leftarrow 2 & & & +2 & 2 \\
 & & & \leftarrow 3 & & & +3 \\
 & & & & \leftarrow 5 & & \\
 & & & & & \leftarrow 8 &
\end{array}
$$

We now know how many elementary paths there are ⓧ to ⓧ of each length 2, 4, 5, 6, 7, 8, 9: subtract 2 and use the above table.

We count all paths by considering elementary paths:

Patterns for length 9: (need an odd length elementary path)

225	252	522	45	54	27	72	9	ⓧ to ⓧ pattern
003	030	300	23	32	05	50	7	look up ⓨ to ⓨ
1·1·1	1·1·1	1·1·1	1·1	1·1	1·3	3·1	8	take products

$1 + 1 + 1 + 1 + 1 + 3 + 3 + 8 = 19$ ✓

One can also figure this out using "generating functions" which is a topic studied in combinatorics.

[3] Find all 2×2 matrices A that satisfy the condition

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ two conditions on four unknowns

$\Rightarrow$ 2-dimensional family of solutions

$$[a\ b]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -2 \quad \Rightarrow \quad [a\ b] = [0\ {-1}] + [2\ {-1}]s$$

$$[c\ d]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \quad \Rightarrow \quad [c\ d] = [1\ 0] + [2\ {-1}]t$$

$$\boxed{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}t}$$

check:

$$\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}t\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}s + \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}t$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}s + \begin{bmatrix} 0 \\ 0 \end{bmatrix}t = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \checkmark$$

This is a conceptual problem with easy arithmetic.

In linear algebra we are trained to recognize a linear system of equations, even (especially) in disguised form.
This problem tests whether one has absorbed this world view.

We then understand that the complete set of solutions is an affine space (point, line, plane...) whose dimension can be found by counting unknowns and independent conditions, and which can be written in the standard form "general = particular + homogeneous".

Giving four matrices that work is giving four points on a plane, and reveals that one never saw this as a linear system of equations.

An answer $\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{2\times2} + \underbrace{\underbrace{\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}}_{2\times2}\underbrace{\begin{bmatrix} s \\ t \end{bmatrix}}_{2\times1}}_{2\times1}$ is absurd because the shapes don't match.
Always check matrix shapes!

[4] Find a system of equations having as solution set the following affine subspace of $\mathbb{R}^4$.

$$V \quad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

V is a plane in $\mathbb{R}^4 \Rightarrow$ need two independent conditions

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

How do I quickly check your answer?

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$? $\quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ columns add to zero?

$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$? $\quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ columns add to zero?

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ -1 \end{bmatrix}$? $\quad \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ columns add to right hand side?

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$V \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} a \end{bmatrix}$$

$$W \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} \right) = \begin{bmatrix} 3 \end{bmatrix}$$

$\Rightarrow$ equ for W is $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \end{bmatrix}$

plug in parametrization for V:

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} a \end{bmatrix} \right) = \begin{bmatrix} 3 \end{bmatrix}$$

$$3 + 0a = 3$$

So a can be any value $\Rightarrow$ $\boxed{V \text{ is contained in } W}$

Check:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \end{bmatrix} / 3$$

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} / 3$$