Exam 2, 8:40am

Linear Algebra, Dave Bayer, November 4, 2021

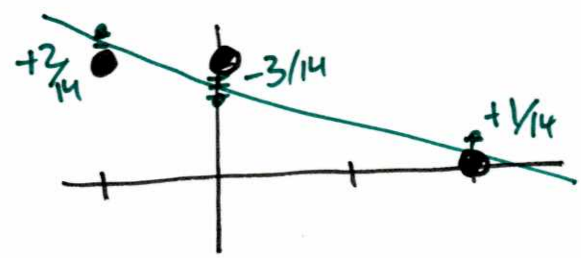
Name: Answers **Uni:**

[1]	[2]	[3]	[4]	[5]	Total
6	6	6	6	6	30

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\boxed{y = -\frac{5}{14} x + \frac{11}{14}}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 5 \end{bmatrix} \Big/ 14 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \end{bmatrix} \Big/ 14$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{check:}} = \begin{bmatrix} 16 \\ 11 \\ 1 \end{bmatrix} \Big/ 14 - \begin{bmatrix} 14 \\ 14 \\ 0 \end{bmatrix} \Big/ 14 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \Big/ 14 \perp \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} ✓$$

[2] Find the orthogonal projection of the vector $(1, 0, 0, 0)$ onto the subspace of $\mathbb{R}^4$ spanned by the vectors $(1, 1, 1, 1)$ and $(0, 1, 2, 1)$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 4 \end{bmatrix} /8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} /4$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} /4 = \boxed{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} /4}$$

$$\underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix} /4}_{\text{check:}} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} /4 \quad \perp \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

[3] Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 4 \end{vmatrix} = \underset{13}{\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix}} \underset{+}{+} \underset{4}{\begin{vmatrix} 1 & -1 \\ 0 & 4 \end{vmatrix}} = \boxed{17}$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 4 \end{vmatrix} = \underset{13}{\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix}} \underset{+}{-} \underset{4}{\begin{vmatrix} -1 & 0 \\ 1 & 4 \end{vmatrix}} = 17$$

$$\begin{vmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 4 \end{vmatrix} = \underset{1}{\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}} \underset{+}{+} \underset{4 \cdot 4}{4\begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix}} = 17$$

[4] Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & -9 \\ 1 & -1 & -2 \\ -2 & -1 & 4 \end{bmatrix} \Big/ {-3}$$

$$\begin{array}{c|cccc} 2 & 0 & 1 & 2 & 0 \\ \hline 1 & 2 & 1 & 1 & 2 \\ 5 & 1 & 2 & 5 & 1 \\ 2 & 0 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 & 2 \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} -3 & -3 & 9 \\ -1 & 1 & 2 \\ 2 & 1 & -4 \end{bmatrix} \Big/ 3}$$

check:

$$\begin{bmatrix} -3 & -3 & 9 \\ -1 & 1 & 2 \\ 2 & 1 & -4 \end{bmatrix} \Big/ 3 \begin{bmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \Big/ 3 \quad ✓$$

[5] Consider $\mathbb{R}^3$ equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Find an orthogonal basis for $\mathbb{R}^3$ with respect to this inner product.

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow V_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}}$$

Check:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \checkmark$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0 \checkmark$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 0 \checkmark$$