Exam 2, 10:10am

Linear Algebra, Dave Bayer, November 4, 2021

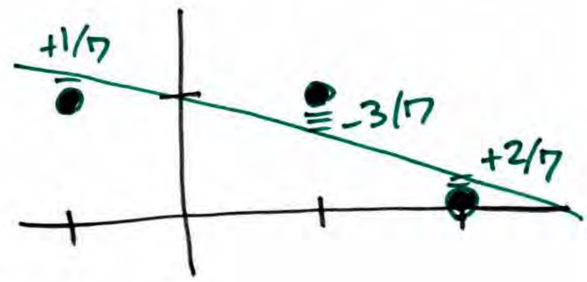
Name: Answers **Uni:**

[1]	[2]	[3]	[4]	[5]	Total
6	6	6	6	6	30

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$



$$\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\boxed{y = -\tfrac{2}{7}x + \tfrac{6}{7}}$$

$$-\begin{bmatrix} 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix} / 7$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{check:}} = \begin{bmatrix} 8 \\ 4 \\ 2 \end{bmatrix} / 7 - \begin{bmatrix} 7 \\ 7 \\ 0 \end{bmatrix} / 7 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} / 7 \perp \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

[2] Find the orthogonal projection of the vector $(1,1,0,0)$ onto the subspace of $\mathbb{R}^4$ spanned by the vectors $(1,1,1,0)$ and $(0,1,1,1)$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{1}{5}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} / 5$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} / 5 = \boxed{\begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} / 5}$$

$$\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix} / 5}_{\text{check}} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix} / 5 \perp \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad ✓$$

[3] Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 0 & 1 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \underbrace{2\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}}_{2 \cdot 10} + \underbrace{\begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix}}_{3} = \boxed{23}$$

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \underbrace{2\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}}_{2 \cdot 10} + \underbrace{\begin{vmatrix} -1 & 0 \\ 1 & 3 \end{vmatrix}}_{3} = 23$$

$$\begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & -1 \\ 0 & 1 & 3 \end{vmatrix} = \underbrace{\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix}}_{2} + \underbrace{3\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}_{3 \cdot 7} = 23$$

[4] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ -3 & 1 & 2 \\ 3 & -1 & -5 \end{bmatrix} / -3$$

$$\begin{array}{c|cccc|} 1 & 3 & 0 & 1 & 3 \\ \hline 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 0 & 1 & 3 \\ 2 & 1 & 1 & 2 & 1 \\ \hline \end{array}$$

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & -2 \\ -3 & 1 & 5 \end{bmatrix} / 3}$$

check:

$$\begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & -2 \\ -3 & 1 & 5 \end{bmatrix} / 3 \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} / 3 \quad ✓$$

[5] Consider $\mathbb{R}^3$ equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Find an orthogonal basis for $\mathbb{R}^3$ with respect to this inner product.

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow V_3 = \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}$$

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} \right\}}$$

check:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0 \checkmark$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = 0 \checkmark$$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix} = 0 \checkmark$$