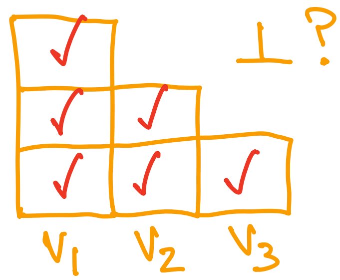
Final Exam

Linear Algebra, Dave Bayer, December 17-23, 2020

[1] Find an orthogonal basis for $\mathbb{R}^4$ that includes the vector $(1, 0, 0, 1)$.

$$\boxed{\begin{aligned} v_1 &= (1,0,0,1) \\ v_2 &= (1,0,0,-1) \\ v_3 &= (0,1,0,0) \\ v_4 &= (0,0,1,0) \end{aligned}}$$



(There are many correct answers.)

[2] Find a system of equations having as solution set the image of the following map from $\mathbb{R}^3$ to $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \\ -1 & -2 & -1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$2 = 1 + 3$

so rank 2

need 2 conditions

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 0 & 2 \\ 1 & -1 \\ -1 & -1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$* \boxed{\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

(There are many correct answers.)

[3] Let V be the subspace of $\mathbb{R}^4$ defined by the system of equations

②=①+③
so 2 conditions

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Find the 4×4 matrix A that projects $\mathbb{R}^4$ orthogonally onto V.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

choose ⊥ vectors so we can add projections

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix} / 2 + \begin{bmatrix} -2 \\ 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 & -2 \end{bmatrix} / 10$$

4	-2	-2	4
-2	5 1	-5 1	-2
-2	-5 1	5 1	-2
4	-2	-2	4

/10

$$\boxed{\begin{bmatrix} 2 & -1 & -1 & 2 \\ -1 & 3 & -2 & -1 \\ -1 & -2 & 3 & -1 \\ 2 & -1 & -1 & 2 \end{bmatrix} / 5}$$

check: ☑

$$\begin{bmatrix} 2 & -1 & -1 & 2 \\ -1 & 3 & -2 & -1 \\ -1 & -2 & 3 & -1 \\ 2 & -1 & -1 & 2 \end{bmatrix} / 5 \begin{bmatrix} 0 & -2 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -10 & 0 & 0 \\ 5 & 5 & 0 & 0 \\ -5 & 5 & 0 & 0 \\ 0 & -10 & 0 & 0 \end{bmatrix} / 5$$

preserve V

send equations to zero

[4] Find A^n where A is the matrix

$$A = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\boxed{\lambda = -1, 3 \qquad A^n = \frac{(-1)^n}{4} \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} + \frac{3^n}{4} \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}}$$

$$\begin{aligned} r+s &= \operatorname{trace}(A) = 2 \\ rs &= \det(A) = -3 \end{aligned} \implies r, s = -1, 3$$

λ	w	$A - \lambda I$	v	projection
-1	$\begin{bmatrix} 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$	$\begin{bmatrix} 3 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} / 4 = \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} / 4$
3	$\begin{bmatrix} 1 & 3 \end{bmatrix}$	$\begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} / 4 = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} / 4$

$$* \quad \boxed{A^n = (-1)^n \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} / 4 + 3^n \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} / 4}$$

check . .

$$I = A^0 = (-1)^0 \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} / 4 + 3^0 \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} / 4 = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} / 4$$

$n=0$

$$\begin{array}{cc|cc} 3 & 1 & -3 & 3 \\ \hline -1 & 1 & 1 & 3 \end{array} / 4 \quad \checkmark$$

$$A = A^1 = (-1)^1 \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} / 4 + 3^1 \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} / 4 = \begin{bmatrix} 0 & 12 \\ 4 & 8 \end{bmatrix} / 4$$

$n=1$

$$\begin{array}{cc|cc} -3 & 3 & 3 & 9 \\ \hline 1 & 3 & -1 & 9 \end{array} / 4 \quad \checkmark$$

[5] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\boxed{\lambda = 0, 1, 3 \qquad e^{At} = \frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{bmatrix} + \frac{e^t}{2}\begin{bmatrix} 0 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{bmatrix} + \frac{e^{3t}}{6}\begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 4 \end{bmatrix}}$$

A is singular $\Rightarrow$ 0 is an eigenvalue

$A-1$ has zero row $\Rightarrow$ 1 is an eigenvalue

$\mathrm{trace}(A) = 4 = 0+1+3 \Rightarrow$ 3 is an eigenvalue

λ	w	$A-\lambda I$	v	projection
0	$[2\ 0\ {-1}]$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}[2\ 0\ {-1}]/3 = \begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}/6$
1	$[0\ 1\ 0]$	$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 2 & 2 & 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}[0\ 1\ 0]/2 = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 6 & 0 \\ 0 & -6 & 0 \end{bmatrix}/6$
3	$[2\ 3\ 2]$	$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -2 & 0 \\ 2 & 2 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}[2\ 3\ 2]/6 = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 4 \end{bmatrix}/6$

$$e^{f(A)} = f(0)\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}/6 + f(1)\begin{bmatrix} 0 & -3 & 0 \\ 0 & 6 & 0 \\ 0 & -6 & 0 \end{bmatrix}/6 + f(3)\begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 4 \end{bmatrix}/6$$

$f(x) = e^{xt}$:

$$\boxed{e^{At} = 1\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}/6 + e^t\begin{bmatrix} 0 & -3 & 0 \\ 0 & 6 & 0 \\ 0 & -6 & 0 \end{bmatrix}/6 + e^{3t}\begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 4 \end{bmatrix}/6}$$

check $f(x)=1$:

$$I = 1\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}/6 + 1\begin{bmatrix} 0 & -3 & 0 \\ 0 & 6 & 0 \\ 0 & -6 & 0 \end{bmatrix}/6 + 1\begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 4 \end{bmatrix}/6 = \begin{bmatrix} 6 & & \\ & 6 & \\ & & 6 \end{bmatrix}/6$$

check $f(x)=x$:

$$A = 0\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -4 & 0 & 2 \end{bmatrix}/6 + 1\begin{bmatrix} 0 & -3 & 0 \\ 0 & 6 & 0 \\ 0 & -6 & 0 \end{bmatrix}/6 + 3\begin{bmatrix} 2 & 3 & 2 \\ 0 & 0 & 0 \\ 4 & 6 & 4 \end{bmatrix}/6 = \begin{bmatrix} 6 & 6 & 6 \\ 0 & 6 & 0 \\ 12 & 12 & 12 \end{bmatrix}/6$$

$I =$

4 2	-3 3	-2 2
	6	
-4 4	-6 6	2 4

/6 ✓

$A =$

6	-3 9	6
	6	
12	-6 18	12

/6 ✓

[5] Form of the answer?

$$e^{At} = 1\begin{bmatrix}2&0&-1\\0&0&0\\-2&0&1\end{bmatrix}/3 + e^{t}\begin{bmatrix}0&-1&0\\0&2&0\\0&-2&0\end{bmatrix}/2 + e^{3t}\begin{bmatrix}2&3&2\\0&0&0\\4&6&4\end{bmatrix}/6$$

This is the cleanest looking conventional answer.
Each matrix projects onto an eigenspace.
One can spot by eye that without coefficients they add to I.

$$e^{At} = 1\begin{bmatrix}4&0&-2\\0&0&0\\-4&0&2\end{bmatrix}/6 + e^{t}\begin{bmatrix}0&-3&0\\0&6&0\\0&-6&0\end{bmatrix}/6 + e^{3t}\begin{bmatrix}2&3&2\\0&0&0\\4&6&4\end{bmatrix}/6$$

Putting the matrices over a common denominator makes them easier to check.

$$e^{At} = 1\begin{bmatrix}1\\0\\-1\end{bmatrix}\begin{bmatrix}2&0&-1\end{bmatrix}/3 + e^{t}\begin{bmatrix}-1\\2\\-2\end{bmatrix}\begin{bmatrix}0&1&0\end{bmatrix}/2 + e^{3t}\begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}2&3&2\end{bmatrix}/6$$

This form is efficient and beautiful, making the structure clear at a glance.

$$e^{At} = 1\begin{bmatrix}-4&0&2\\0&0&0\\4&0&-2\end{bmatrix}/-6 + e^{t}\begin{bmatrix}0&3&0\\0&-6&0\\0&6&0\end{bmatrix}/-6 + e^{3t}\begin{bmatrix}-2&-3&-2\\0&0&0\\-4&-6&-4\end{bmatrix}/-6$$

Reall 7. Accepted, but almost marked incomplete.

$$e^{At} = \begin{bmatrix}2/3+e^{3t}/3 & -e^{t}/2+e^{3t}/2 & -1/3+e^{3t}/3\\ 0 & e^{t} & 0\\ -2/3+2e^{3t}/3 & -e^{t}+e^{3t} & 1/3+2e^{3t}/3\end{bmatrix}$$

ouch!

Technically correct, but very hard to understand, check, or grade.

When I see this form in a book, it tells me the author is going through the motions but doesn't care about the answer. Usually a student gives this answer after change of coordinates, because they didn't learn to read off the individual matrices. I may suffer reading this; I feel bad that someone had to write it.

[6] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lambda = 2, 2 \qquad e^{At} = e^{2t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \qquad y = e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

Eigenvalues r, s $\quad r+s = \operatorname{trace}(A) = 4$, $\; rs = \det(A) = 4 \;\Rightarrow\; r, s = 2, 2$

Use Jordan canonical form as a model for computing in original coords:

$$J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}}_{①} = \underbrace{2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{②} + \underbrace{\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}}_{③ \text{ "}\varepsilon\text{"}}$$

$$\begin{array}{cc|c} 4 & -2 & -1 \\ \hline 4 & & -2 \end{array}$$

③ = ① − ②

Check $\varepsilon^2 = 0$: $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ✓

$$e^{(2+\varepsilon)t} = e^{2t}e^{\varepsilon t} = e^{2t}(1+\varepsilon t) = e^{2t} + te^{2t}\varepsilon \quad \text{if } \varepsilon^2 = 0$$

$$e^{At} = e^{(2I+\varepsilon)t} = e^{2t}I + te^{2t}\varepsilon \quad \text{as above, } \varepsilon = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

$$y = e^{At}y(0) = \left(e^{2t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$* \quad \boxed{y = e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} -1 \\ -2 \end{bmatrix}}$$

check:

$$y' = 2e^{2t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (e^{2t} + 2te^{2t})\begin{bmatrix} -1 \\ -2 \end{bmatrix} = e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix} + te^{2t}\begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

$$Ay = e^{2t}\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 4 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix} + te^{2t}\begin{bmatrix} -2 \\ -4 \end{bmatrix}$$

✓

[7] Express the quadratic form

$$3x^2 + 3y^2 + 2xz - 2yz + 2z^2$$

as a sum of squares of orthogonal linear forms.

$$\lambda = 1, 3, 4 \qquad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\frac{1}{6}(x - y - 2z)^2 + \frac{3}{2}(x + y)^2 + \frac{4}{3}(x - y + z)^2$$

$$3x^2+3y^2+2xz-2yz+2z^2 = [x\ y\ z]\underbrace{\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & -1 \\ 1 & -1 & 2 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Eigenvalues r, s, t

$r+s+t = \operatorname{trace}(A) = 8$

$rst = \det(A) = 3\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = 15 - 3 = 12$

$r = 3$ is clearly an eigenvalue

$\Rightarrow \begin{matrix} s+t = 5 \\ st = 4 \end{matrix} \Rightarrow s, t = 1, 4$ $\qquad \boxed{r, s, t = 1, 3, 4}$

λ	w	$A - \lambda I$	v	projection	$[x\ y\ z]\, A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
1	$[1\ {-1}\ {-2}]$	$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}[1\ {-1}\ {-2}]/6$	$\frac{1}{6}\left([x\ y\ z]\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\right)\left([1\ {-1}\ {-2}]\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$
3	$[1\ 1\ 0]$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}[1\ 1\ 0]/2$	$+\frac{3}{2}\left([x\ y\ z]\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)\left([1\ 1\ 0]\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$
4	$[1\ {-1}\ 1]$	$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & -1 & -2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}[1\ {-1}\ 1]/3$	$+\frac{4}{3}\left([x\ y\ z]\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right)\left([1\ {-1}\ 1]\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

check: $\left.\begin{array}{c|c|c} 1\ 3\ 2 & -1\ 3\ {-2} & -2\ \ 2 \\ \hline -1\ 3\ {-2} & 1\ 3\ 2 & 2\ {-2} \\ \hline -2\ \ \ 2 & 2\ {-2} & 4\ \ 2 \end{array}\right. /6 = I$ ✓

$$\boxed{\tfrac{1}{6}(x-y-2z)^2 + \tfrac{3}{2}(x+y)^2 + \tfrac{4}{3}(x-y+z)^2}$$

[7] check:

$$\begin{array}{c|ccc} * & x & y & z \\ \hline x & x^2 & xy & xz \\ y & xy & y^2 & yz \\ z & xz & yz & z^2 \end{array}$$

$$\tfrac{1}{6}(x-y-2z)^2 + \tfrac{3}{2}(x+y)^2 + \tfrac{4}{3}(x-y+z)^2$$

$$\begin{array}{c|ccc} * & 1 & -1 & -2 \\ \hline 1 & 1 & -1 & -2 \\ -1 & -1 & 1 & 2 \\ -2 & -2 & 2 & 4 \end{array} /6 \;+\; \begin{array}{c|ccc} * & 1 & 1 & \\ \hline 1 & 1 & 1 & \\ 1 & 1 & 1 & \\ & & & \end{array} 9 /6 \;+\; \begin{array}{c|ccc} * & 1 & -1 & 1 \\ \hline 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{array} 8 /6 \;=\; \begin{array}{|c|c|c|} \hline 18 & & 6 \\ \hline & 18 & -6 \\ \hline 6 & -6 & 12 \\ \hline \end{array} /6$$

$$\begin{array}{ccc|ccc|cc} 1 & 9 & 8 & -1 & 9 & -8 & -2 & 8 \\ \hline -1 & 9 & -8 & 1 & 9 & 8 & 2 & -8 \\ \hline -2 & & 8 & 2 & & -8 & 4 & 8 \end{array} \;=\; \begin{array}{|c|c|c|} \hline 3 & & 1 \\ \hline & 3 & -1 \\ \hline 1 & -1 & 2 \\ \hline \end{array}$$

$$3x^2 + 3y^2 + 2xz - 2yz + 2z^2$$

☑

[8] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\begin{bmatrix}\ \end{bmatrix} \quad \begin{bmatrix} 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -4 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -4 & 0 & 3 & 1 \\ 0 & -4 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ -4 & 0 & 3 & 1 & 0 \\ 0 & -4 & 0 & 3 & 1 \\ 0 & 0 & -4 & 0 & 3 \end{bmatrix}$$

$$1 \qquad 3 \qquad 9 \qquad 23 \qquad 57 \qquad 135$$

For example, $f(4) = 57$ and $f(5) = 135$.
Find a recurrence relation for $f(n)$.
Using Jordan canonical form, solve this recurrence relation to find a closed form formula for $f(n)$.

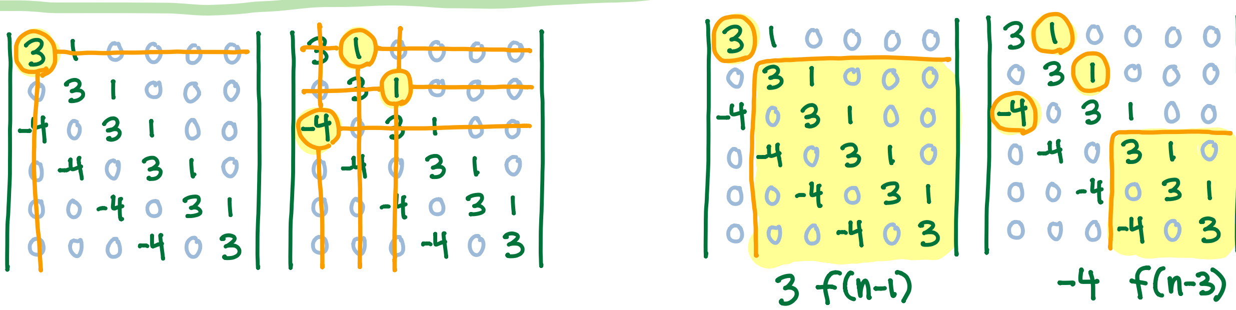
$$f(0) = 1, \quad f(1) = 3, \quad f(2) = 9, \quad f(n) \;=\; 3\,f(n-1) \;-\; 4\,f(n-3)$$

$$\begin{bmatrix} f(3) \\ f(2) \\ f(1) \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f(2) \\ f(1) \\ f(0) \end{bmatrix}, \qquad \begin{bmatrix} f(n+2) \\ f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} f(2) \\ f(1) \\ f(0) \end{bmatrix}$$

$$f(n) \;=\; \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix} \;=\; \frac{1}{9}\,(-1)^n \;+\; \frac{8}{9}\,2^n \;+\; \frac{4}{3}\,n\,2^{n-1}$$

$$f(4) \;=\; \frac{1}{9}\,(-1)^4 \;+\; \frac{8}{9}\,2^4 \;+\; \frac{4}{3}\,4 \cdot 2^3 \;=\; (1 + 8 \cdot 16 + 12 \cdot 4 \cdot 8)/9 \;=\; (1 + 128 + 384)/9 \;=\; 57$$

$$f(5) \;=\; \frac{1}{9}\,(-1)^5 \;+\; \frac{8}{9}\,2^5 \;+\; \frac{4}{3}\,5 \cdot 2^4 \;=\; (-1 + 8 \cdot 32 + 12 \cdot 5 \cdot 16)/9 \;=\; (-1 + 256 + 960)/9 \;=\; 135$$



Each term of the determinant either begins on the diagonal, or as an off-diagonal triple. So recursively,

$$\boxed{f(n) = 3f(n-1) - 4f(n-3)}$$

check:

n	0	1	2	3	4	5
f(n)	1	3	9	23	57	135
-3f(n-1)		-3	-9	-27	-69	-171
4f(n-3)				4	12	36
0				0	0	0

[8] $\begin{bmatrix} f(3) \\ f(2) \\ f(1) \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f(2) \\ f(1) \\ f(0) \end{bmatrix} \Rightarrow \begin{bmatrix} f(n+2) \\ f(n+1) \\ f(n) \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} f(2) \\ f(1) \\ f(0) \end{bmatrix}$

$$\Rightarrow \boxed{f(n) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix}}$$

$A = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $\quad -|A-\lambda I| = \lambda^3 - \text{trace}(A)\lambda^2 + \text{huh}(A)\lambda - \det(A)$

$$\boxed{\begin{aligned} \text{trace}(A) &= 3+0+0 = 3 \\ \text{huh}(A) &= \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0 \\ \det(A) &= 1 \cdot 1 \cdot (-4) = -4 \end{aligned}}$$

λ	-4	-2	-1	1	2	4
λ^3	-64	-8	-1	1	8	64
$-3\lambda^2$	-48	-12	-3	-3	-12	-48
4	4	4	4	4	4	4
			0		0	

$$-|A-\lambda I| = \lambda^3 - 3\lambda^2 + 4 = 0$$
$$(\lambda+1)(\lambda-2)^2 = 0$$

$-1+2+2=3$ $\quad \boxed{\lambda = -1, 2, 2}$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^n = (-1)^n \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + n2^{n-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^n = (-1)^n \underbrace{\begin{bmatrix} 1 & -4 & 4 \\ -1 & 4 & -4 \\ 1 & -4 & 4 \end{bmatrix} / 9}_{①} + 2^n \underbrace{\begin{bmatrix} 8 & 4 & -4 \\ 1 & 5 & 4 \\ -1 & 4 & 5 \end{bmatrix} / 9}_{②} + n2^{n-1} \underbrace{\begin{bmatrix} 4 & -4 & -8 \\ 2 & -2 & -4 \\ 1 & -1 & -2 \end{bmatrix} / 3}_{③}$$

$① \Rightarrow ② \Rightarrow ③$

① Find projection matrix for $\lambda = -1$ as usual.

w	$A-\lambda I$	v	projection
$\begin{bmatrix} 1 & -4 & 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 0 & -4 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 4 \end{bmatrix} / 9 = \begin{bmatrix} 1 & -4 & 4 \\ -1 & 4 & -4 \\ 1 & -4 & 4 \end{bmatrix} / 9$

② $= I - ①$ $\quad \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} / 9 - \begin{bmatrix} 1 & -4 & 4 \\ -1 & 4 & -4 \\ 1 & -4 & 4 \end{bmatrix} / 9 = \begin{bmatrix} 8 & 4 & -4 \\ 1 & 5 & 4 \\ -1 & 4 & 5 \end{bmatrix} / 9$

$(n=0)$

[8] ③ $= A +$ ① $- 2$ ② $\quad (n=1)$

$$\begin{bmatrix}3&0&-4\\1&0&0\\0&1&0\end{bmatrix} - (-1)\begin{bmatrix}1&-4&4\\-1&4&-4\\1&-4&4\end{bmatrix}\Big/9 - 2\begin{bmatrix}8&4&-4\\1&5&4\\-1&4&5\end{bmatrix}\Big/9$$

27 1 −16	−4 −8	−36 4 8
9 −1 −2	4 −16	−4 −8
1 2	9 −4 −8	4 −10

/9

$$= \begin{bmatrix}12&-12&-24\\6&-6&-12\\3&-3&-6\end{bmatrix}\Big/9 = \underbrace{\begin{bmatrix}4&-4&-8\\2&-2&-4\\1&-1&-2\end{bmatrix}\Big/3}_{\text{check: squares to zero } \checkmark}$$

$$f(n) = \begin{bmatrix}0&0&1\end{bmatrix}\begin{bmatrix}3&0&-4\\1&0&0\\0&1&0\end{bmatrix}^n\begin{bmatrix}9\\3\\1\end{bmatrix}$$

$$= \begin{bmatrix}0&0&1\end{bmatrix}\left((-1)^n\begin{bmatrix}1&-4&4\\-1&4&-4\\1&-4&4\end{bmatrix}\Big/9 + 2^n\begin{bmatrix}8&4&-4\\1&5&4\\-1&4&5\end{bmatrix}\Big/9 + n2^{n-1}\begin{bmatrix}4&-4&-8\\2&-2&-4\\1&-1&-2\end{bmatrix}\Big/3\right)\begin{bmatrix}9\\3\\1\end{bmatrix}$$

$$= \left((-1)^n\begin{bmatrix}1&-4&4\end{bmatrix}/9 + 2^n\begin{bmatrix}-1&4&5\end{bmatrix}/9 + n2^{n-1}\begin{bmatrix}1&-1&-2\end{bmatrix}/3\right)\begin{bmatrix}9\\3\\1\end{bmatrix}$$

$$\boxed{f(n) = \frac{1}{9}(-1)^n + \frac{8}{9}2^n + \frac{4}{3}n2^{n-1}}$$

check:

n	$f(n)$	$9f(n) = (-1)^n + 8\cdot 2^n + 12n2^{n-1}$						
0	1	9	1	+	8	+	0	☑
1	3	27	−1	+	16	+	12	☑
2	9	81	1	+	32	+	48	☑
3	23	207	−1	+	64	+	144	☑
4	57	513	1	+	128	+	384	☑
5	135	1215	−1	+	256	+	960	☑