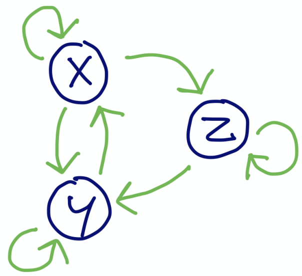
Exam 1

Linear Algebra, Dave Bayer, October 2-5, 2020

[1] Using matrix multiplication, count the number of paths of length six from x to y.

$\boxed{\text{86 paths}}$



$$\begin{array}{c|ccc} & & \text{in} & \\ & x & y & z \\ \hline x & 1 & 1 & 0 \\ y & 1 & 1 & 1 \\ z & 1 & 0 & 1 \\ \text{out} & & & \end{array}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}}_{A^2}$$

$$\underbrace{\begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}}_{A^2} \underbrace{\begin{bmatrix} 2 & & \\ 3 & & \\ 2 & & \end{bmatrix}}_{A^2} = \underbrace{\begin{bmatrix} 12 & & \\ 16 & & \\ 9 & & \end{bmatrix}}_{A^4}$$

$$\underbrace{\begin{bmatrix} & & \\ 3 & 2 & 2 \\ & & \end{bmatrix}}_{A^2} \underbrace{\begin{bmatrix} 12 & & \\ 16 & & \\ 9 & & \end{bmatrix}}_{A^4} = \underbrace{\begin{array}{c|ccc} & x & & \\ \hline & & & \\ y & 86 & & \\ & & & \end{array}}_{A^6}$$

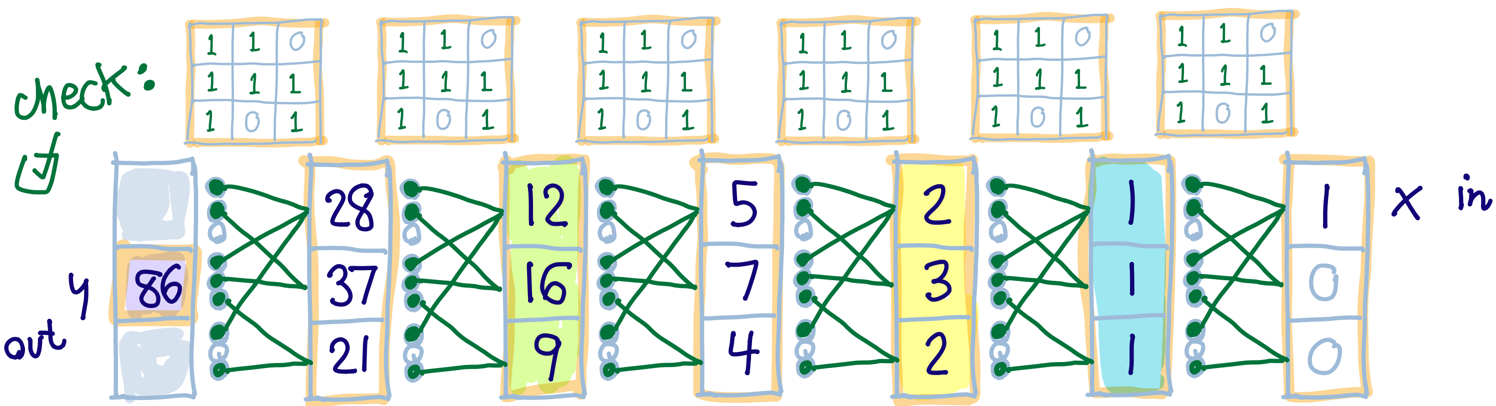
2	2	1	
2	3	2	
4	6	2	12

3	2	2	
2	3	2	
6	6	4	16

2	1	1	
2	3	2	
4	3	2	9

3	2	2	
12	16	9	
36	32	18	86

check: ✓



[2] Find all solutions to the system of equations

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \\ \cancel{0} & \cancel{4} & \cancel{4} & \cancel{4} \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \cancel{4} \end{bmatrix}$$

4 dimensions w, x, y, z

-2 conditions ①, ②

2 dimensional solution

① and ② are independent

Third equation ③ is sum of first two

$$\left.\begin{bmatrix} 1 \\ -1 \end{bmatrix}\boxed{} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\boxed{} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\boxed{1} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\boxed{} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\} \text{particular solution}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}\boxed{} + \underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\boxed{1} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\boxed{-2} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\boxed{1}}_{\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] \text{ is average of } \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\boxed{1} + \begin{bmatrix} 1 \\ 3 \end{bmatrix}\boxed{1} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}\boxed{-1}}_{\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] \text{ is sum of } \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}\boxed{} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

two independent homogeneous solutions

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}$$

check

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & 3 & 2 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ -2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix}$$

[3] Find a system of equations having as solution set the image of the following map from $\mathbb{R}^3$ to $\mathbb{R}^4$.

4 dimensions w,x,y,z
−2 conditions
2 dimensional solution set

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

middle column is sum of outside cols so ima is 2-dimensional

want to find system of equations: $\begin{bmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

step 1: find rows that dot to zero with homogeneous solutions

step 2: compute right hand side

$$\boxed{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

[4] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

plug in and solve

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}}$$

the intersection is a point

check:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \checkmark$$

check:

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \checkmark$$

This point is in both spaces

[5] Express $\begin{bmatrix} s \\ t \end{bmatrix}$ in terms of $\begin{bmatrix} a \\ b \end{bmatrix}$ for the following parametrizations of the same affine subspace of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \qquad \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 6 & 9 \\ 4 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{array}{cccc} \begin{bmatrix} a \\ b \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 \\ 7 \\ 6 \\ 5 \end{bmatrix} & \begin{bmatrix} 2 \\ 10 \\ 8 \\ 6 \end{bmatrix} \\ \hline \begin{bmatrix} s \\ t \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad t=1$$

$$\begin{bmatrix} 2 \\ 7 \\ 6 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 6 \\ 3 \end{bmatrix} \quad t=3$$

$$\begin{bmatrix} 2 \\ 10 \\ 8 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 8 \\ 4 \end{bmatrix} \quad t=4$$

$$\begin{array}{l} \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} s \\ t \end{bmatrix} \\ \hline \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{array}$$

answer: $\boxed{\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

check
see if
plugging back in
works?

$$\begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 3 \\ 0 & 2 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 6 & 9 \\ 4 & 6 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

✓

second method: set parametrizations equal to each other, and solve

$$\begin{bmatrix}0\\-2\\0\\2\end{bmatrix}+\begin{bmatrix}1&0\\0&3\\0&2\\0&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}w\\x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\1\\2\\3\end{bmatrix}+\begin{bmatrix}1&1\\6&9\\4&6\\2&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&3\\0&2\\0&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}-\begin{bmatrix}1&1\\6&9\\4&6\\2&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\1\\2\\3\end{bmatrix}-\begin{bmatrix}0\\-2\\0\\2\end{bmatrix}$$

$$\left[\begin{array}{cc|cc}1&0&1&1\\0&3&6&9\\0&2&4&6\\0&1&2&3\end{array}\right]\begin{bmatrix}s\\t\\-a\\-b\end{bmatrix}=\begin{bmatrix}1\\3\\2\\1\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}1&0&1&1&1\\0&3&6&9&3\\0&2&4&6&2\\0&1&2&3&1\end{array}\right]\xrightarrow[\substack{2\leftarrow 2-3(4)\\ 3\leftarrow 3-2(4)}]{}\left[\begin{array}{cccc|c}1&0&1&1&1\\ \cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}\\ \cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}&\cancel{0}\\0&1&2&3&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|c}1&0&1&1&1\\0&1&2&3&1\end{array}\right]$$

$$\left[\begin{array}{cc|cc}1&0&1&1\\0&1&2&3\end{array}\right]\begin{bmatrix}s\\t\\-a\\-b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}s\\t\end{bmatrix}-\begin{bmatrix}1&1\\2&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

answer:

$$\boxed{\begin{bmatrix}s\\t\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1&1\\2&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}}$$