F17 8:40 Exam 2 Problem 1
Linear Algebra, Dave Bayer

test1a2p1

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Test 1

Name Answer Key Uni

[1] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 3 & 0 \end{bmatrix}$$

$$y = \boxed{-1/5}\, x + \boxed{2/5}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 11 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{4}{35} & -\frac{3}{35} \\ -\frac{3}{35} & \frac{11}{35} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \end{pmatrix}$$

check:

x	y	ax+b	5Δ
-1	1	3/5	2
0	0	2/5	-2
1	0	1/5	-1
3	0	-1/5	1

$\perp$ to

1	-1
1	0
1	1
1	3
✓	✓

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Test 1

[2] Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 3 & -1 & -1 \\ -1 & -2 & 5 \\ -1 & 5 & -2 \end{bmatrix}$$

(Do not write a negative denominator.)

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Inverse[{{3, 1, 1}, {1, 1, 2}, {1, 2, 1}}] = {{3, -1, -1}, {-1, -2, 5}, {-1, 5, -2}}/7
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$$\begin{array}{c|cccc} 3 & 1 & 1 & 3 & 1 \\ \hline 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 3 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{array}$$

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & 2 & -5 \\ 1 & -5 & 2 \end{bmatrix} / -7$$

so fix signs

Test 1

[3] Consider $\mathbb{R}^3$ equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle \;=\; \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Using this inner product, find the orthogonal projection of the vector $(2,2,2)$ onto the plane spanned by $(1,0,1)$ and $(0,1,1)$.

$$\begin{bmatrix} 2 & 1 & 3 \end{bmatrix}$$

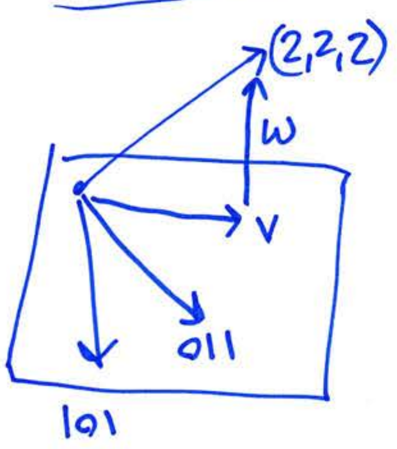
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$A^TAx = A^Tb$ solves usual projection
$A^TBAx = A^TBb$ solves for inner product using B



$(2,2,2) = v + w$ v in plane, answer
$w \perp$ to plane

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \quad \Big\} \text{ find } w \perp \text{ to plane}$$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0 \quad \text{so } w = t(0,1,-1)$$

want $(2,2,2) - t(0,1,-1)$ in plane $x+y=z$
$4 - t = 2 + t \qquad t = 1$

$$\Rightarrow \boxed{v = (2,1,3)}$$

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Test 1

[4] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\begin{bmatrix} 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 3 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & 3 & 1 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Find $f(1)$ and $f(2)$. Find a recurrence relation for $f(n)$. Find $f(6)$.

$f(1) = 1 \qquad f(2) = 4$

$f(n) = (1)\, f(n-1) + (3)\, f(n-2)$

$f(6) = 97$

$$\text{Det}[(1)] = 1$$

$$\text{Det}\left[\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}\right] = 4$$

$$\text{Det}\left[\begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & -1 \\ 0 & 3 & 1 \end{pmatrix}\right] = 7$$

$$f(n) = (1) f(n-1) + (3) f(n-2)$$

$$\text{Det}\left[\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 3 & 1 & -1 & 0 & 0 & 0 \\ 0 & 3 & 1 & -1 & 0 & 0 \\ 0 & 0 & 3 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & 0 & 3 & 1 \end{pmatrix}\right] = 97$$

n	f(n)
1	1
2	4
3	7
4	19
5	40
6	$\boxed{97}$

×3
×1

×3
×1

⋮

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Test 1

[5] Find a system of eigenvalues and eigenvectors for the matrix A, where

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

$$\lambda_1,\ \lambda_2 = \boxed{3},\ \boxed{4}$$

$$v_1,\ v_2 = \begin{bmatrix} \boxed{1} \\ \boxed{1} \end{bmatrix},\ \begin{bmatrix} \boxed{2} \\ \boxed{1} \end{bmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ 1 & 2 \end{pmatrix}$$

$$12 - 7x + x^2 = 0$$

$$\begin{pmatrix} 4 & 3 \\ \{2, 1\} & \{1, 1\} \end{pmatrix}$$

Check

$$\begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 1 \end{bmatrix} \checkmark$$