F17 8:40 Exam 1 Problem 1
Linear Algebra, Dave Bayer

test1a1p1

[Reserved for Score]

Test 1

Name Solutions Uni

[1] Find the general solution to the following system of equations.

$$\begin{array}{c} ① \\ ② \\ ③ \end{array}\begin{bmatrix} 0 & 2 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 4 \\ 0 & 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$$

③ = ① + ②

①, ② independent

5 vars − 2 conditions = 3 dimensional solution

$$\boxed{\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}}$$

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -4 \\ 0 & -3 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

Map $\mathbb{R}^5 \to \mathbb{R}^3$. Rows ① and ② are not multiples of each other, so linearly independent.
Row ③ is sum of rows ①, ② so redundant.

2 conditions on 5 variables leaves a 5−2=3 dimensional solution space, so answer must be of the form

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ * \\ * \end{bmatrix} + \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

3 parameters for 3 dimensions

3 columns for 3 parameters.

This analysis is the most important part of the problem.
If your answer doesn't have this shape, it gives the impression that you don't understand the problem at all.

[9|p|]...

Homogeneous solutions, particular solution

$$\begin{bmatrix} 0 & \cancel{2}\,2 & 1 & 1 & -4 \\ 0 & -1 & 1 & 0 & 4 \\ 0 & 1 & 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}}_{\text{particular}} \text{ or } \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\text{homogeneous}}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$$

$\begin{bmatrix} 0 \\ 0 \\ 5 \\ -2 \\ 0 \end{bmatrix}$: $\quad \overset{(5)}{\begin{bmatrix} 5 \\ 5 \\ 10 \end{bmatrix}} + \overset{(-2)}{\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ ✓

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$: $\quad \overset{(1)}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$\begin{bmatrix} 0 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}$: $\quad \overset{(1)}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}} + \overset{(1)}{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}} + \overset{(-3)}{\begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

$\begin{bmatrix} 0 \\ 0 \\ -4 \\ 5 \\ 1 \end{bmatrix}$: $\quad \overset{(-4)}{\begin{bmatrix} -4 \\ -4 \\ -8 \end{bmatrix}} + \overset{(5)}{\begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}} + \overset{(1)}{\begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

It is essential that our basic solutions are independent. Take two multiples of the same vector (such as negative) is a wrong answer.

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Test 1

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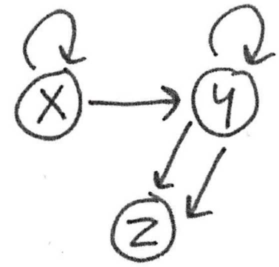
test1a1p2

[2] Using matrix multiplication, count the number of paths of length eight from x to z.

$$\begin{array}{cc} & \text{from} \\ & \begin{array}{ccc} x & y & z \end{array} \\ \text{to} \begin{array}{c} x \\ y \\ z \end{array} & \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}}_{A} \end{array}$$



number of paths = 14

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number of paths = 14

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & & \\ 6 & 2 & 0 \end{bmatrix}$$

$$A^8 = \underbrace{\begin{bmatrix} & & \\ & & \\ 6 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & & \\ 4 & & \\ 6 & & \end{bmatrix}}_{\text{working backwards, need these entries}} = \begin{array}{cc} & x \\ z & \begin{bmatrix} & & \\ & & \\ \boxed{14} & & \end{bmatrix} \end{array}$$

need this entry

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Linear Algebra, Dave Bayer

test1a1p3

[Reserved for Score]

Test 1

[3] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q \\ r \end{bmatrix}}_{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}}, \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

$$\begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$$S = 1$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t$$

Recheck: $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1+t \\ 1 \end{bmatrix}}_{\text{belongs to left}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \end{bmatrix}}_{\text{belongs to } \cancel{\text{taite}} \text{ right}}$

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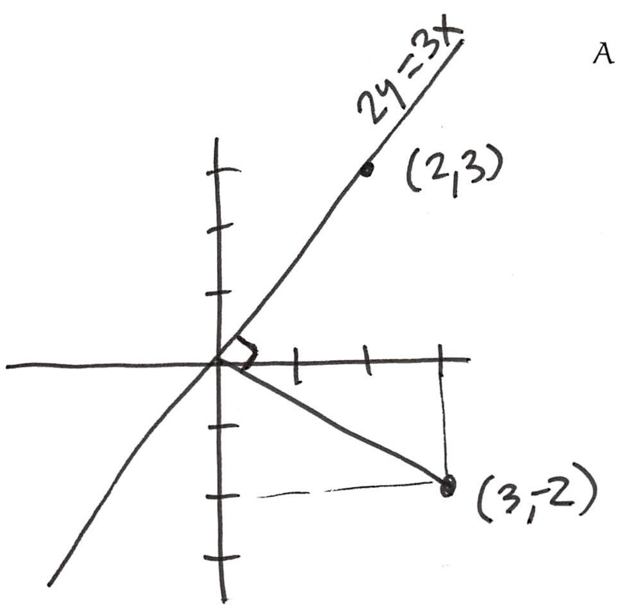
Test 1

[4] Find the 2×2 matrix A that reflects across the line $2y = 3x$.

$$A = \frac{1}{\square}\begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

$$A = \frac{1}{13}\begin{bmatrix} -5 & 12 \\ 12 & 5 \end{bmatrix}$$



$(2,3) \mapsto (2,3)$
$(3,-2) \mapsto (-3,2)$

$$A\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} +2 & +3 \\ +3 & -2 \end{bmatrix}\Big/13$$

$$= \begin{bmatrix} -5 & 12 \\ 12 & 5 \end{bmatrix}\Big/13$$

check: $\underbrace{\begin{bmatrix} -5 & 12 \\ 12 & 5 \end{bmatrix}\Big/13}_{A} \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 26 & -39 \\ 39 & 26 \end{bmatrix}\Big/13 = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$ ✓

testla1p5

Test 1

[5] Find a basis for the subspace of $\mathbb{R}^5$ spanned by the following vectors:

$$\underset{a}{(1,1,0,0,2)},\quad \underset{b}{(1,0,1,0,2)},\quad \underset{c}{(1,0,0,1,2)},\quad \underset{d}{(0,1,1,0,2)},\quad \underset{e}{(0,1,0,1,2)},\quad \underset{f}{(0,0,1,1,2)}$$

$$\left.\begin{aligned} c-e &= (1,-1,0,0,0) \\ a-b &= (0,1,-1,0,0) \\ d-e &= (0,0,1,-1,0) \end{aligned}\right\}$$

using these vectors we can slide entries around at will in the first four slots, turning for example any of a, b, c, d, e, or f into

$$(0,0,0,2,2)$$

These four vectors are linearly independent because they start in different positions.

The last entry is the sum of first four.
In an equation, (v,w,x,y,z) satisfies $v+w+x+y=z$
so the subspace cannot be all of $\mathbb{R}^5$. It has dimension 4.

Thus a basis is

$$\boxed{\begin{matrix}(1,-1,0,0,0)\\(0,1,-1,0,0)\\(0,0,1,-1,0)\\(0,0,0,2,2)\end{matrix}}$$

Note this problem does not ask to extend to all of $\mathbb{R}^5$.