F16 10:10 Exam 1 Problem 1
Linear Algebra, Dave Bayer

test1a2p1

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Test 1

Name Solutions Uni

[1] Find the determinant of the matrix

$$\boxed{\begin{matrix} + & & & & \\ - & & & & \\ + & - & + & - & + \\ - & & & & \\ + & & & & \end{matrix}}$$

$$A = \begin{bmatrix} 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 4 \end{bmatrix}$$

$\det(A) = \boxed{72}$

$$\det(A) = -3 \begin{vmatrix} 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 0 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = -3(-4)6 = 12 \cdot 6 = 72$$

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Linear Algebra, Dave Bayer

test1a2p2

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Test 1

[2] Find the inverse to the matrix

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{c|cccc} 2 & 3 & 1 & 2 & 3 \\ \hline 3 & 2 & 1 & 3 & 2 \\ 4 & 1 & 0 & 4 & 1 \\ 2 & 3 & 1 & 2 & 3 \\ 3 & 2 & 1 & 3 & 2 \end{array}$$

$$\begin{bmatrix} -1 & 4 & -5 \\ 1 & -4 & 10 \\ 1 & 1 & -5 \end{bmatrix} / 5$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} -1 & 4 & -5 \\ 1 & -4 & 10 \\ 1 & 1 & -5 \end{bmatrix}$$

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test1b2p3

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Test 1

[3] Using Cramer's rule, solve for y in the system of equations

$$\begin{bmatrix} a & 1 & 2 \\ b & 1 & 1 \\ c & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$y = \frac{(-1)\,a + (6)\,b + (-1)\,c}{(3)\,a + (-2)\,b + (-1)\,c}$$

$$\frac{\begin{vmatrix} a & 1 & 2 \\ b & 1 & 1 \\ c & 5 & 4 \end{vmatrix}}{\begin{vmatrix} a & 1 & 2 \\ b & 1 & 1 \\ c & 1 & 4 \end{vmatrix}} = \frac{a\begin{vmatrix} 1 & 1 \\ 5 & 4 \end{vmatrix} - b\begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix} + c\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}{a\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} - b\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} + c\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}$$

$$= \frac{-a + 6b - c}{3a - 2b - c}$$

check:

$$\begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 1 \\ 6 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$$

$$\left.\begin{array}{l} a=2 \\ b=2 \\ c=6 \end{array}\right\} \Rightarrow y = -1 \qquad \frac{-2+12-6}{6-4-6} = -1 \ \checkmark$$

(choose values for a,b,c where we know the solu)

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test1a2p4

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Test 1

[4] Find a system of eigenvalues and eigenvectors for the matrix A, and find a formula for A^n, where

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\lambda_1 + \lambda_2 = 4+5 = 9$$

$$\lambda_1 \lambda_2 = 4 \cdot 5 - 2 \cdot 1 = 18$$

$$\boxed{3, 6}$$

$$\lambda_1,\ \lambda_2 = 3,\ 6$$

$$v_1,\ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A^n = \frac{3^n}{3} \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} + \frac{6^n}{3} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$3: \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$6: \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$$A^n = 3^n \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix} / 3 + 6^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} / 3$$

Check: $n=0$, I: $\begin{array}{cc|cc} 2 & & -1 & \\ & 1 & & 1 \\ \hline -2 & & 1 & \\ & 2 & & 2 \end{array}$ $/3$ ✓

$n=1$, A: $\begin{array}{cc|cc} 6 & & -3 & \\ & 6 & & 6 \\ \hline -6 & & 3 & \\ & 12 & & 12 \end{array}$ $/3$ ✓

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Test 1

[5] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$[\,] \quad [4] \quad \begin{bmatrix} 4 & 1 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 0 \\ 3 & 4 & 1 \\ 0 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 3 & 4 & 1 & 0 & 0 \\ 0 & 3 & 4 & 1 & 0 \\ 0 & 0 & 3 & 4 & 1 \\ 0 & 0 & 0 & 3 & 4 \end{bmatrix}$$

Find a recurrence relation for $f(n)$. Express $f(n)$ using a matrix power. Find a formula for $f(n)$.

$$f(n) = (4)\, f(n-1) + (-3)\, f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$f(n) = \frac{-1}{2} + \frac{3}{2}(3)^n$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

$\lambda_1 + \lambda_2 = 4$

$\lambda_1 \lambda_2 = +3$

$\boxed{1,3}$

$$A - I: \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$A - 3I: \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0$$

$$A^n = (1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \end{bmatrix} / 2 + (3)^n \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} / 2$$

$$A^n \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \frac{-1}{2}(1)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3}{2}(3)^n \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\boxed{f(n) = \frac{-1}{2}(1)^n + \frac{3}{2}(3)^n}$$