F16 8:40 Exam 1 Problem 1
Linear Algebra, Dave Bayer

test1a1p1

[Reserved for Score]

Test 1

Name (Solutions) Uni ______

[1] Find the general solution to the following system of equations.

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix} 1 & -1 & \\ & 1 & \\ -1 & -1 & 1 \end{bmatrix}}_{\text{valid in one step because this matrix is nonsingular} \iff \text{reduces to } I} \quad \begin{matrix} ① - ② \\ ② \\ ③ - ① - ② \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 2 & -1 \\ 1 & 1 & 0 & 1 & 3 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} 3 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-1) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} (-1) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-2) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 2 & 1 & 1 & 3 \\ 1 & 1 & 0 & 1 \\ 3 & 2 & 1 & 4 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 3 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ -1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) \stackrel{?}{=} \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \quad \boxed{\checkmark}$$

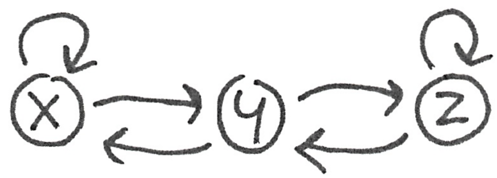
F16 8:40 Exam 1 Problem 2
Linear Algebra, Dave Bayer

test1a1p2

[Reserved for Score]

Test 1

[2] Using matrix multiplication, count the number of paths of length eight from y to y.



$$A = \begin{array}{c|ccc} & x & y & z \text{ (from)} \\ \hline x & 1 & 1 & 0 \\ y & 1 & 0 & 1 \\ z & 0 & 1 & 1 \\ \text{(to)} & & & \end{array}$$

number of paths = 86

$$A^2 = \underset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^4 = \underset{A^2}{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}} \underset{A^2}{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}} = \begin{bmatrix} & 5 & \\ 5 & 6 & 5 \\ & 5 & \end{bmatrix}$$

$$A^8 = \underset{A^4}{\begin{bmatrix} & & \\ \boxed{5\ 6\ 5} & & \\ & & \end{bmatrix}} \underset{A^4}{\begin{bmatrix} & 5 & \\ & 6 & \\ & 5 & \end{bmatrix}} = \begin{bmatrix} & & \\ & \boxed{86} & \\ & & \end{bmatrix}$$

$5 \cdot 5 + 6 \cdot 6 + 5 \cdot 5 = 86$

only compute what we need

check: Make a left or right loop of ≥ 2 steps before returning to y. Then any path of remaining length.

$$\Longleftrightarrow \quad f(n) = 2(f(n-2) + f(n-3) + f(n-4) + \ldots)$$

n	0	1	2	3	4	5	6	7	8
$f(n)$	1	0	$\boxed{2}$	2	$\boxed{6}$	10	22	42	$\boxed{86}$
Σ	1	1	3	5	11	21	43		

x2 x2 x2

✓

Test 1

[3] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}, \qquad x - 2y + z = 2$$

plug parametrization into equation.

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t}$$

(answer can take many forms)

$$\begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right) = 2$$

$$0 + \begin{bmatrix} -2 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = 2$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} t \right)$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} t$$

check: In plane $x - 2y + z = 2$? $\qquad 2 + 0t = 2$ ✓

Test 1

[4] By least squares, find the equation of the form $z = ax + by$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(Note that b is multiplied by y in this equation.)

$$z = \boxed{2/3}\, x + \boxed{2/3}\, y$$

want $\left\{ \begin{aligned} 1 &= a\cdot 1 + b\cdot 0 \\ 1 &= a\cdot 0 + b\cdot 1 \\ 1 &= a\cdot 1 + b\cdot 1 \end{aligned} \right\}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$ can't solve exactly

least squares: $Ax = b \Rightarrow A^TAx = A^Tb$ instead

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

recall $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ so $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}/3$

check

x	y	z	$2/3x + 2/3y$	Δ
1	0	1	2/3	1/3
0	1	1	2/3	1/3
1	1	1	4/3	-1/3

$\times 3$

and $(1,1,-1)\cdot(1,0,1) = 0$
$\cdot(0,1,1) = 0$

✓

check that difference is $\perp$ to image of A

Test 1

[5] Let V be the vector space $\mathbb{R}^3$, equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Using this inner product to define orthogonality, find the orthogonal projection of the vector $(0,0,1)$ onto the plane defined by the equation

$$x - y + z = 0$$

$$\boxed{\begin{bmatrix} -4 & 3 & 7 \end{bmatrix} / 6}$$

Need to move $\perp$ to plane. What direction is this?

Plane $(1,-1,1)\cdot(x,y,z) = 0$

Two vectors in plane: $(1,1,0)$, $(0,1,1)$ basis for plane

Find vector $\perp$ to these:

$(4,-3,-1)$ is $\perp$ to plane

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

①+② $3\ 4\ 0$

Now $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix} t \right) = 0$

$1 + 6t = 0 \Rightarrow t = -1/6$

$(0,0,1) - 1/6\,(4,-3,-1)$

$= [(0,0,6) - (4,-3,-1)]/6 = (-4,3,7)/6$