F16 10:10 Exam 1 Problem 1
Linear Algebra, Dave Bayer

test1b1p1

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Test 1

Name (Solutions) Uni

[1] Find the general solution to the following system of equations.

$$\begin{bmatrix} 5 & 7 & 1 & 2 \\ 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{array}{c} 1-2-3 \\ 2-3 \\ 3 \end{array} \left[\begin{array}{cccc|c} \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 0 & 1 & 1 \end{array}\right]$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}}$$

valid in one step because this matrix is nonsingular $\Longleftrightarrow$ reduces to I

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} 1 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} 1 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} 1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} 3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 5 & 7 & 1 & 2 \\ 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) \stackrel{?}{=} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

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Linear Algebra, Dave Bayer

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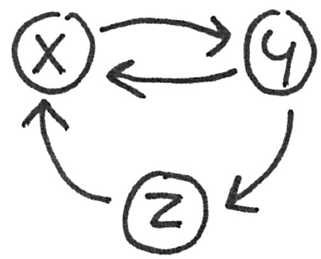
Test 1

[2] Using matrix multiplication, count the number of paths of length 16 from x to x.



$$A = \begin{array}{c|ccc|l} & x & y & z & \text{in} \\ x & 0 & 1 & 1 & \\ y & 1 & 0 & 0 & \\ z & 0 & 1 & 0 & \\ \text{out} & & & & \end{array}$$

number of paths = 37

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^8 = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A^4}\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A^4} = \begin{bmatrix} \boxed{4} & \boxed{5} & \boxed{3} \\ \boxed{3} & & \\ \boxed{2} & & \end{bmatrix}$$

checked

$$A^{16} = \underbrace{\begin{bmatrix} \boxed{4\ 5\ 3} & & \\ & & \\ & & \end{bmatrix}}_{A^8}\underbrace{\begin{bmatrix} \boxed{4} & & \\ \boxed{3} & & \\ \boxed{2} & & \end{bmatrix}}_{A^8} = \begin{bmatrix} \boxed{37} & & \\ & & \\ & & \end{bmatrix}$$

only compute what we need

$$\begin{array}{ccc} 4 & 5 & 3 \\ 4 & 3 & 2 \\ \hline 16 & 15 & 6 \end{array}$$

check: Mix of loops length 2 and 3.

$\Leftrightarrow$ $f(n) = f(n-2) + f(n-3)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$f(n)$	1	0	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37

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Test 1

[3] Find the intersection of the following two affine subspaces of $\mathbb{R}^3$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}, \qquad 2x + y - z = 2$$

plug parametrization into equation

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t}$$

(answer may take many forms)

$$\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix}\right) = 2$$

$$-1 + \begin{bmatrix} 3 & -3 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = 2$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \end{bmatrix} = 1$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t\right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t$$

check: In plane $2x+y-z=2$? $\quad 2 + 0t = 2$ ✓

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Test 1

[4] By least squares, find the equation of the form $y = ax^2 + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$$

(Note that x is *squared* in $y = ax^2 + b$.)

$y = (1/2)\,x^2 + 1$

want $\left\{\begin{aligned} 1 &= a\cdot 1 + b \\ 1 &= a\cdot 0 + b \\ 2 &= a\cdot 1 + b \end{aligned}\right\}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$ can't solve exactly

least squares: $Ax=b \Rightarrow A^TAx = A^Tb$ instead

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix}(1/2) + \begin{bmatrix} 2 \\ 3 \end{bmatrix}1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \text{so} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

check

1	x^2	y	$\frac{1}{2}x^2+1$	Δ
1	1	1	3/2	1/2
1	0	1	1	0
1	1	2	3/2	−1/2

×2

and $\underbrace{\begin{aligned}(1,0,-1)\cdot(1,1,1) &= 0 \\ \cdot(1,0,1) &= 0\end{aligned}}_{\text{check that difference is } \perp \text{ to image of } A}$

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Test 1

[5] Let V be the vector space $\mathbb{R}^3$, equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Using this inner product to define orthogonality, find an orthogonal basis for the plane defined by the equation

$$x + y = 0$$

Extend this basis to an orthogonal basis for $\mathbb{R}^3$.

$$v_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$
$$v_3 = \begin{bmatrix} 3 & 2 & 2 \end{bmatrix}$$

Take $v_1 = (0,0,1)$

want $v_2 \perp v_1$, and in plane

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $v_2 = (-1,1,1)$

Now need $v_3 \perp v_1, v_2$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 3 & 0 \end{bmatrix} \boxed{\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_3 = (3,2,2)$

Check $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓