



Dave Bayer (instructor)

[1] Solve the following system of equations.

independent $\rightarrow$

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

rank 2:
row 2 = $\frac{\text{row 1} + \text{row 3}}{2}$
(coverage)

dim count: $4 - 2 = 2$ dim homog solus

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

find particular solution: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

find two independent homogeneous solutions:

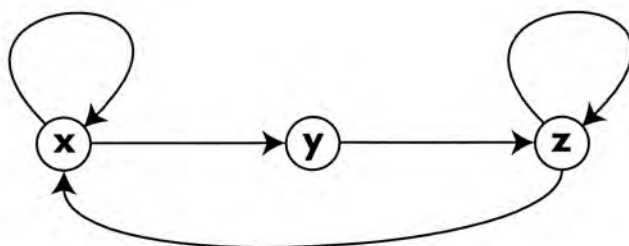
$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} 3 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$



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[2] Using matrix multiplication, count the number of paths of length nine from x to z.



number of paths = 49

49 paths

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad A^9 = \begin{bmatrix} 65 & 49 & 86 \\ 37 & 28 & 49 \\ 49 & 37 & 65 \end{bmatrix}$$

to

	from x	from y	from z
x	1	0	1
y	1	0	0
z	0	1	1

A

$$A^2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 4 & 3 & 5 \\ 2 & 2 & 3 \\ 3 & 2 & 4 \end{bmatrix}$$

3) finally add on another row 3
1) row 1 + row 3 of A^2
2) now add on row 2

$$A^8 = \begin{bmatrix} \text{shaded} \\ 21 \\ 28 \\ \text{shaded} \end{bmatrix}$$

We only need two entries
 $(2,2,3) \cdot (4,2,3) = 8+4+9 = 21$
 $(3,2,4) \cdot (4,2,3) = 12+4+12 = 28$

$$A^9 = \begin{bmatrix} \text{shaded} \\ 49 \\ \text{shaded} \end{bmatrix}$$



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[3] Express A as a product of three elementary matrices, where

$$A = \begin{bmatrix} 7 & 1 \\ 4 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 1 \\ 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 7 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} \nwarrow & \nwarrow & \nwarrow \\ \textcircled{2} \leftarrow 4\textcircled{2} & \textcircled{1} \leftarrow \textcircled{1} + 7\textcircled{2} & \textcircled{1} \leftrightarrow \textcircled{2} \end{array}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}}_{\begin{bmatrix} 7 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 7 & 1 \\ 4 & 0 \end{bmatrix}$$

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db89h1p4

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[4] Find the matrix A such that

$$A \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 4 \\ 3 & 3 & 1 \end{bmatrix}$$

column reduce

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ \hline 1 & 3 & 5 \\ 3 & 4 & 4 \\ 3 & 3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 2 & 5 \\ 3 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 2 & 1 \\ 3 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$\textcircled{2} \leftarrow \textcircled{2} - \textcircled{1}$ $\textcircled{3} \leftarrow \textcircled{3} - 2\textcircled{2}$ $\textcircled{1} \leftarrow \textcircled{1} - \textcircled{3}$ sort columns

check: $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 4 \\ 3 & 3 & 1 \end{bmatrix} \quad \checkmark$

Or transpose, row reduce, transpose back.
Exactly the same numbers, in more familiar positions.



db89h1p5

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[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 1 \end{bmatrix}$$



Dave Bayer (instructor)

[6] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} p$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} t$$

Set equal to each other:

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ -2 & 1 & -3 & 1 \\ 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ -t \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

rank 3, 1 dim soln

$$\begin{bmatrix} r \\ s \\ -t \\ -u \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} p \Rightarrow \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} p$$

(ignore t, u not needed)

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} p = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} p$$

check: use other way, ignore r, s

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -3 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} p = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} p$$

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