F15 Exam 2 Problem 1
Linear Algebra, Dave Bayer

test1a2p1

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Test 1

Name Solutions Uni

[1] Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

v	w	$v+w$	$2v+w$	$v+2w$	$2v+2w$
$(1,1,0,1)$	$(1,0,1,1)$	$(2,1,1,2)$	$(3,2,1,3)$	$(3,1,2,3)$	$(4,2,2,4)$

Extend this basis to a basis for $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

v, w are a basis for the plane V.

$(1,0,0,0)$, $(0,0,0,1)$ extends this basis to a basis for $\mathbb{R}^4$:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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F15 Exam 2 Problem 2
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Test 1

[2] Find the 3×3 matrix A such that

$$A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad A\begin{bmatrix}1\\-2\\0\end{bmatrix}=\begin{bmatrix}1\\-2\\0\end{bmatrix},\quad A\begin{bmatrix}0\\1\\-2\end{bmatrix}=\begin{bmatrix}0\\1\\-2\end{bmatrix}$$

$$\boxed{A=\frac{1}{7}\begin{bmatrix}3&-2&-1\\-4&5&-1\\-4&-2&6\end{bmatrix}}$$

$A = I - B$ where

$$B\begin{bmatrix}1&1&0\\1&-2&1\\1&0&-2\end{bmatrix}=\begin{bmatrix}1&0&0\\1&0&0\\1&0&0\end{bmatrix}$$

(1,1,1) (1,-2,0) (0,1,-2)
is multiplied by...

A	0	1	1
+ B	1	0	0
I	1	1	1

B vanishes on plane $4x+2y+z=0$
B maps (1,1,1) to itself.

$$B=\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}4&2&1\end{bmatrix}\Big/7 \qquad 7=(1,1,1)\cdot(4,2,1)$$

$$=\begin{bmatrix}4&2&1\\4&2&1\\4&2&1\end{bmatrix}\Big/7$$

$$\Rightarrow A=\left(\begin{bmatrix}7&&\\&7&\\&&7\end{bmatrix}-\begin{bmatrix}4&2&1\\4&2&1\\4&2&1\end{bmatrix}\right)\Big/7=\begin{bmatrix}3&-2&-1\\-4&5&-1\\-4&-2&6\end{bmatrix}\Big/7$$

check:

$$\begin{bmatrix}3&-2&-1\\-4&5&-1\\-4&-2&6\end{bmatrix}\Big/7\begin{bmatrix}1&1&0\\1&-2&1\\1&0&-2\end{bmatrix}=\begin{bmatrix}0&7&0\\0&-14&7\\0&0&-14\end{bmatrix}\Big/7=\begin{bmatrix}0&1&0\\0&-2&1\\0&0&-2\end{bmatrix}$$ ✓

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Test 1

[3] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\left\{\begin{aligned} b &= 1 \\ a+b &= 1 \\ 3a+b &= 2 \end{aligned}\right\} \xleftarrow{\text{want}} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$y = \boxed{5/14}\; x + \boxed{12/14}$$

$$\Downarrow \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

but can't solve. Replace $Ax=b$ by $A^TAx = A^Tb$:

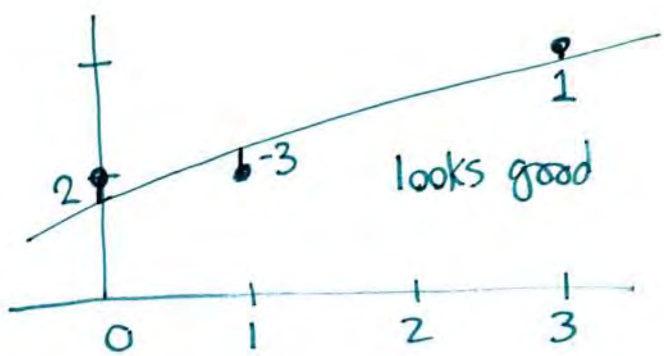
$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -4 & 10 \end{bmatrix}\Big/14 \begin{bmatrix} 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}\Big/14$$

$$y = 5/14\, x + 12/14$$

1	x_i	y_i	$5/14\, x_i + 12/14$	14Δ
1	0	1	12/14	2
1	1	1	17/14	-3
1	3	2	27/14	1

dots to zero with columns of A ☑



F15 Exam 2 Problem 4
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Test 1

[4] Find the 4×4 matrix that projects orthogonally onto the plane spanned by the vectors $(1,0,1,0)$ and $(0,1,0,1)$.

$$A = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$(1,0,1,0)$ and $(0,1,0,1)$ are $\perp$, so we can sum the projections to each direction.

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}\begin{bmatrix}1&0&1&0\end{bmatrix}/2 + \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\begin{bmatrix}0&1&0&1\end{bmatrix}/2$$

$$= \begin{bmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{bmatrix}/2 + \begin{bmatrix}0&0&0&0\\0&1&0&1\\0&0&0&0\\0&1&0&1\end{bmatrix}/2 = \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}/2$$

check:

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}/2 \; \begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix} \quad ✓$$

in plane, $\perp$ to plane, fixed, zero

F15 Exam 2 Problem 5
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Test 1

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[5] Let V be the vector space $\mathbb{R}^3$, equipped with the inner product

$$\langle (a,b,c),(d,e,f)\rangle = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = 2ad + ae + bd + 2be + bf + ce + 2cf$$

Using this inner product rather than the dot product, find the orthogonal projection of the vector $(1,0,0)$ onto the plane spanned by $(0,1,0)$ and $(0,0,1)$.

$$\boxed{\begin{bmatrix} 0 & 2 & -1 \end{bmatrix}/3}$$

Find $(a,b,c) \perp$ to both $(0,1,0)$, $(0,0,1)$:

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = a + 2b + c = 0$$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = b + 2c = 0$$

$$(\cancel{}, 2, -1)$$

$$(a,b,c) = (-3, 2, -1)$$

Now

$$(1,0,0) = \underbrace{t(-3,2,-1)}_{\perp \text{ to plane}} + \underbrace{\left[(1,0,0) - t(-3,2,-1)\right]}_{\perp \text{ projection into plane}}$$

$t = -1/3$ zeros x coordinate

$$(1,0,0) = \underbrace{(1, -2/3, 1/3)}_{\perp \text{ to plane}} + \underbrace{(0, 2/3, -1/3)}_{\text{in plane}}$$

☑ sum ok

check:

$$\begin{bmatrix} 3 & -2 & 1 \end{bmatrix}/3 \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}/3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \end{bmatrix} \checkmark$$