F14 Practice 4 Problem 1
Linear Algebra, Dave Bayer

[1] Find A^n where A is the matrix

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$$

$$\boxed{A^n = \frac{(-1)^n}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \frac{(2)^n}{3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}}$$

$$\boxed{\lambda = -1, 2 \qquad A^n = \frac{(-1)^n}{3} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} + \frac{2^n}{3} \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}}$$

(computer's solution from program that found this problem)

$r+s = 1$, $rs = -2 \Rightarrow r, s = -1, 2$

$-1 \quad [1\ 2]\begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}[1\ 2]/3 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}/3$

$2 \quad [1\ -1]\begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix}[1\ -1]/3 = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}/3$

(left eigenvector)$(A-\lambda)$(right eigenvector)

$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}/3 = I$ ✓

$-(A-\lambda_2)$ ✓
$(A-\lambda_1)$ ✓
$(\lambda_2-\lambda_1)$ ✓
where $\lambda_1 < \lambda_2$

check: does what I wrote <u>in the box</u> work for $n=1$?

$$\left.\begin{array}{cc|cc} -1 & 4 & -2 & -4 \\ \hline -1 & -2 & -2 & 2 \end{array}\right/3 = \begin{bmatrix} 3 & -6 \\ -3 & 0 \end{bmatrix}/3 = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} \stackrel{?}{=} A \ \boxed{✓}$$

F14 Practice 4 Problem 2
Linear Algebra, Dave Bayer

[2] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 0 \end{bmatrix}$$

$$\boxed{e^{At} = \frac{e^{-2t}}{5}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} + \frac{e^{3t}}{5}\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}}$$

$$\boxed{\lambda = -2, 3 \qquad e^{At} = \frac{e^{-2t}}{5}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} + \frac{e^{3t}}{5}\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}}$$

sum = 1   −2, 3
prod = −6

$-2 \quad [2\ 3]\boxed{\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}[2\ 3]/5 = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}/5 \qquad -(A-3)$ ✓

$(A+2)$ ✓

$3 \quad [1\ -1]\boxed{\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix}}\begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ -2 \end{bmatrix}[1\ -1]/5 = \begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix}/5 \qquad 3-(-2) = 5$ ✓

$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}/5 = I$ ✓

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check answer in box

$$\frac{d}{dt}e^{At}\Big|_{t=1} = \frac{-2}{5}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \qquad \begin{array}{c|c} -4 \;\; 9 & -6 \;\; -9 \\ \hline -4 \;\; -6 & -6 \;\; 6 \end{array} /5 = \begin{bmatrix} 5 & -15 \\ -10 & 0 \end{bmatrix}/5$$

$= A$ ✓

F14 Practice 4 Problem 3
Linear Algebra, Dave Bayer

[3] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\boxed{y = \frac{e^t}{1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \cancel{\frac{}{}\begin{bmatrix} \\ \end{bmatrix}}}$$

$$\lambda = 1, 2 \qquad e^{At} = e^t \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + e^{2t}\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \qquad y = e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\text{sum} = 3$ $\quad 1, 2$
$\text{prod} = 2$

$1 \quad \begin{bmatrix} -1 & 1 \end{bmatrix} \quad \boxed{\begin{matrix} 2 & -1 \\ 2 & -1 \end{matrix}} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} / 1 = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$ ✓ 5 −

$2 \quad \begin{bmatrix} 2 & -1 \end{bmatrix} \quad \boxed{\begin{matrix} 1 & -1 \\ 2 & -2 \end{matrix}} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \end{bmatrix} / 1 \qquad \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$ ✓ $\quad 1 = 2 - 1$ ✓

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

$$y = e^{At}y(0) = \left(e^t \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t}\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$= e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \cancel{e^{2t}\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

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check: $\quad y' = e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Ay = e^t \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ same ✓

eigenvector for A with $\lambda = 1$

F14 Practice 4 Problem 4
Linear Algebra, Dave Bayer

[4] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

$$\boxed{e^{At} = \frac{e^{-t}}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{te^{-t}}{1} \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}}$$

$$\boxed{\lambda = -1, -1 \qquad e^{At} = e^{-t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t} \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}}$$

sum $= -2$
prod $= -3+4 = 1$ $\quad -1, -1$ $\quad \lambda, \lambda$

repeated root

$A = \lambda I + N$ where $N = A - \lambda I$

$e^{At} = e^{\lambda t} I + te^{\lambda t} N \qquad N^2 = 0$ (2x2 case) (*)

$A = -1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} = A$ ✓

$N^2 = 0$ ✓

$e^{At} = e^{-t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{-t} \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$

check answer in box: $e^{At}\big|_{t=0} = I$? ✓

$\frac{d}{dt} e^{At}\big|_{t=0} = A$?

$$\begin{array}{c|cc} -1 & 2 & -2 \\ \hline & 2 & -1-2 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$ ✓

(*) why? Let $f(x) = \det(A - xI)$
Then $f(A) = 0$

(short reason: A acts like each number λ so $f(\lambda) = 0$, right?)

If $f(x) = (x-\lambda)^2$ (repeated roots λ, λ)
Then $f(A) = (\underbrace{A - \lambda I}_{N})^2 = 0$ so $N^2 = 0$

(This is enough for distinct roots. Watch the movie as roots come together, to see $f(A)$ stays 0.)

Now $e^{At} = e^{\lambda t}(e^{(A-\lambda)t}) = e^{\lambda t}(I + tN)$ because $N^2 = 0$ kills rest of power series.

F14 Practice 4 Problem 5
Linear Algebra, Dave Bayer

[5] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$e^{At} = \frac{e^{-t}}{4}\begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix} + \frac{e^{t}}{2}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^{3t}}{4}\begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix}$$

$\lambda = -1, 1, 3 \qquad e^{At} = \frac{e^{-t}}{4}\begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix} + \frac{e^{t}}{2}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^{3t}}{4}\begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix}$

$r+s+t = 3$ ✓

$rs+rt+st = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1-3+1 = -1$ ✓

$rst = 1\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$ ✓

$|A-\lambda I| = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$

so 1, and eigenvalues of $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

sum = 2, prod = −3   −1, 3

$\boxed{-1, 1, 3}$

$-1 \quad [1\ 0\ {-1}] \quad \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}[1\ 0\ {-1}]/4 = \begin{bmatrix} 2 & 0 & -2 \\ -1 & 0 & 1 \\ -2 & 0 & 2 \end{bmatrix}/4$

$1 \quad [1\ {-2}\ 2] \quad \begin{bmatrix} 0 & 0 & 2 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}[-1\ {+2}\ {-2}]/2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}/2$

$3 \quad [1\ 0\ 1] \quad \begin{bmatrix} -2 & 0 & 2 \\ 2 & -2 & 1 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}[1\ 0\ 1]/4 = \begin{bmatrix} 2 & 0 & 2 \\ 3 & 0 & 3 \\ 2 & 0 & 2 \end{bmatrix}/4$

2 2		−2 2
−1 3 −2	4	1 −4 3
−2 2		2 2

/4 = I ✓

check $\frac{d}{dt}e^{At}\big|_{t=0} = A$?

−2 6		2 6
1 −2 9	4	−1 −4 9
2 6		−2 6

/4 $= \begin{bmatrix} 4 & 0 & 8 \\ 8 & 4 & 4 \\ 8 & 0 & 4 \end{bmatrix}/4$ ✓

F14 Practice 4 Problem 6
Linear Algebra, Dave Bayer

[6] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$e^{At} = \frac{e^{4t}}{9} \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} + \frac{e^{t}}{9} \begin{bmatrix} 5 & -4 & -4 \\ -2 & 7 & -2 \\ -3 & -3 & 6 \end{bmatrix} + \frac{te^{t}}{3} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda = 4, 1, 1$ $\qquad e^{At} = \frac{e^{4t}}{9} \begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} + \frac{e^{t}}{9} \begin{bmatrix} 5 & -4 & -4 \\ -2 & 7 & -2 \\ -3 & -3 & 6 \end{bmatrix} + \frac{te^{t}}{3} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$r+s+t=6$

$rs+rt+st = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 3+2+4 = 9$

$rst = -1\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 2 \cdot 2 = 4$

factors of 4: $-4, -2, -1, 1, 2, 4$ $\qquad$ try $\lambda=1$. $A - \lambda I = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ singular ✓ eigenvalue

$r=1 \Rightarrow s+t=5$

$s+t+st=9 \Rightarrow st=4$ $\quad 1,4 \quad$ $1,1,4$ repeated root 1

$A = 4B + 1C + N$ $\qquad$ Strategy: Find B, then $C = I - B$, then $N = A - 4B - C$.

$I = B + C$

B: $\lambda=4$ $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \boxed{\begin{matrix} -2 & 1 & 2 \\ 1 & -2 & 0 \\ 1 & 1 & -2 \end{matrix}}$ $\cancel{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$ $\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 4 & 4 & 4 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}/9 = B$, $C = \begin{bmatrix} 5 & -4 & -4 \\ -2 & 7 & -2 \\ -3 & -3 & 6 \end{bmatrix}/9$

$N = A - 4B - C$: $\begin{array}{c|c|c} 18-16 & 9-16 & 18-16 \\ -5 & +4 & +4 \\ \hline 9-8 & 18-8 & 0-8 \\ +2 & -7 & +2 \\ \hline 9-12 & 9-12 & 18-12 \\ +3 & +3 & -6 \end{array} /9 = \begin{bmatrix} -3 & -3 & 6 \\ 3 & 3 & -6 \\ 0 & 0 & 0 \end{bmatrix}/9 = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}/3$

checks: $N^2 = 0$ ✓ $\quad N\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} = 0$ ✓

F14 Practice 4 Problem 7
Linear Algebra, Dave Bayer

[7] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{y = \frac{e^{2t}}{1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{te^{2t}}{1}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \cancel{\frac{\square}{\square}\begin{bmatrix} \square \\ \square \\ \square \end{bmatrix}}}$$

$$\lambda = 0, 2, 2 \qquad e^{At} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$y = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} + te^{2t}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

A block triangular $\Rightarrow$ eigenvalues of $2, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad 0, 2, 2$

$I = B + C$

$A = 0B + 2C + N$

$B: \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}/2 = B \quad C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}/2$

$N = A - 2C = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad y = e^{At}y(0)$

checks $N^2 = 0$ ✓
$BN = 0$ ✓

$$y = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{e^{2t}}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{y = e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}$$

check $y' = Ay$? $\quad y' = 2e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (e^{2t} + 2te^{2t})\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

$$= e^{2t}\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} + te^{2t}\begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$$

$$Ay = e^{2t}\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + te^{2t}\begin{bmatrix} 2 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = e^{2t}\begin{bmatrix} 6 \\ 2 \\ 2 \end{bmatrix} + te^{2t}\begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}$$ ✓

F14 Practice 4 Problem 8
Linear Algebra, Dave Bayer

[8] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & 3 & 1 \\ -1 & 2 & -2 \end{bmatrix}, \qquad y(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{y = \frac{e^t}{1}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{te^t}{1}\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \frac{t^2e^t}{1}\begin{bmatrix} -4 \\ 4 \\ 4 \end{bmatrix}}$$

$$\lambda = 1, 1, 1$$

$$e^{At} = e^t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^t\begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & 1 \\ -1 & 2 & -3 \end{bmatrix} + \frac{t^2e^t}{2}\begin{bmatrix} -8 & 0 & -8 \\ 8 & 0 & 8 \\ 8 & 0 & 8 \end{bmatrix}$$

$$y = e^t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + te^t\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \frac{t^2e^t}{2}\begin{bmatrix} -8 \\ 8 \\ 8 \end{bmatrix}$$

$r+s+t = \text{trace}(A) = 2+3-2 = 3$

$rst = \det(A) = 2\underbrace{\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}}_{-8} + 2\underbrace{\begin{vmatrix} 3 & 1 \\ -1 & -2 \end{vmatrix}}_{-5} + 3\underbrace{\begin{vmatrix} 3 & 3 \\ -1 & 2 \end{vmatrix}}_{9} = 2(-8)+2(-5)+3\cdot 9 = 1$

$-16 - 10 + 27$

$-1, 1$ divide 1, try 1: $\quad 1+s+t = 3 \Rightarrow s+t=2 \Rightarrow s,t=1$

$1st = 1 \qquad st=1$

$\begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & 1 \\ -1 & 2 & -3 \end{bmatrix}$ multiples ✓

$\lambda = 1$ is an eigenvalue, $r=1$

$\boxed{\text{1 is eigenvalue with multiplicity 3}}$

$N = A - 1I \qquad N^2 = \begin{bmatrix} -8 & 0 & -8 \\ 8 & 0 & 8 \\ 8 & 0 & 8 \end{bmatrix} \qquad N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ✓ as expected,

$f(\lambda) = \det(A-\lambda I)$

$f(A) = -(A-\lambda I)^3$

$= -N^3 = 0$ ✓

$A = I+N, \; N^3 = 0$

$e^{At} = e^{(I+N)t} = e^t(I + tN + t^2N^2/2)$

$y = (e^tI + te^tN + t^2e^tN^2/2)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$y = e^t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + te^t\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \frac{t^2e^t}{2}\begin{bmatrix} -8 \\ 8 \\ 8 \end{bmatrix}$

-over for check.

check: $y = e^t\begin{bmatrix}0\\1\\1\end{bmatrix} + te^t\begin{bmatrix}1\\3\\-1\end{bmatrix} + t^2e^t\begin{bmatrix}-4\\4\\4\end{bmatrix}$

(practice 4 problem 8 continued)

(copied from answer box so no risk of copying error later.)

$$y' = e^t\begin{bmatrix}0\\1\\1\end{bmatrix} + (e^t+te^t)\begin{bmatrix}1\\3\\-1\end{bmatrix} + (2te^t+t^2e^t)\begin{bmatrix}-4\\4\\4\end{bmatrix}$$

$$= e^t\left(\begin{bmatrix}0\\1\\1\end{bmatrix}+\begin{bmatrix}1\\3\\-1\end{bmatrix}\right) + te^t\left(\begin{bmatrix}1\\3\\-1\end{bmatrix}+\begin{bmatrix}-8\\8\\8\end{bmatrix}\right) + t^2e^t\begin{bmatrix}-4\\4\\4\end{bmatrix}$$

$$= e^t\begin{bmatrix}1\\4\\0\end{bmatrix} + te^t\begin{bmatrix}-7\\11\\7\end{bmatrix} + t^2e^t\begin{bmatrix}-4\\4\\4\end{bmatrix}$$

$$Ay = e^tA\begin{bmatrix}0\\1\\1\end{bmatrix} + te^tA\begin{bmatrix}1\\3\\-1\end{bmatrix} + t^2e^tA\begin{bmatrix}-4\\4\\4\end{bmatrix} = y' \quad \checkmark$$

$$\begin{bmatrix}2&-2&3\\3&3&1\\-1&2&-2\end{bmatrix}\begin{bmatrix}0&1&-4\\1&3&4\\1&-1&4\end{bmatrix} = \begin{bmatrix}1&-7&-4\\4&11&4\\0&7&4\end{bmatrix} \quad \checkmark$$

F14 Practice 4 Problem 9
Linear Algebra, Dave Bayer

[9] Express the quadratic form

$$3x^2 + 3y^2 - 2xz + 2yz + 2z^2$$

as a sum of squares of othogonal linear forms.

$$\boxed{\frac{1}{6}\left(x-y+2z\right)^2 + \frac{3}{2}\left(x+y\right)^2 + \frac{4}{3}\left(x-y-z\right)^2}$$

$$\lambda = 1,3,4 \qquad A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{4}{3}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{6}(x - y + 2z)^2 + \frac{3}{2}(x + y)^2 + \frac{4}{3}(x - y - z)^2$$

$$[x\ y\ z]\underbrace{\begin{bmatrix} 3 & & -1 \\ & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3x^2+3y^2-2xz+2yz+2z^2$$

find eigenvalues for A.

$r+s+t = 3+3+2 = 8$

$rst = 3\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 0 & 3 \\ -1 & 1 \end{vmatrix} = 3\cdot5-3 = 12$

try $r=1$: yes, $A-1I$ is singular,

$\hookrightarrow$ $1+s+t=7$, $1st=12$ $\Rightarrow s,t = 3,4$ $\quad\boxed{1,3,4}$

$$1\quad \boxed{\begin{matrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{matrix}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$3\quad \boxed{\begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{matrix}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$4\quad \boxed{\begin{matrix} -1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & -2 \end{matrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

(symmetric $\Rightarrow$ left and right eigenvectors are same.)

$$A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}/6 + 3\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}/2 + 4\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \end{bmatrix}/3$$

$$\Rightarrow [x\ y\ z]A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}(x-y+2z)^2 + \frac{3}{2}(x+y)^2 + \frac{4}{3}(x-y-z)^2$$

check: (x6)

	x^2	y^2	z^2	xy	xz	yz
$(x-y+2z)^2$	1	1	4	−2	4	−4
$9(x+y)^2$	9	9	0	18	0	0
$+\ 8(x-y-z)^2$	8	8	8	−16	−16	16
	18	18	12	0	−12	12
$\div 6 \hookrightarrow$	3	3	2	0	−2	2

✓