F14 Practice 3 Problem 1
Linear Algebra, Dave Bayer

[1] Find the determinant of the matrix

expand by minors

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 3 & 4 & 1 & 5 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$-1 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -1\Big(\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{0} - \underbrace{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}_{-1}\Big) = -1$$

det(A) = $\boxed{-1}$

check: subtract col 2 from col 4, then col 1 from col 2

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 3 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

one term to determinant, 5 swaps to $I \Rightarrow \det = -1$

Instructions:

1. After you are done and have checked your answer, copy answer into provided form.
2. Show work to support your answer. Without supporting work, a correct answer will receive a score of zero. I do not need all details, but I need a level of detail that could not have easily been copied from a neighbor. I am sorry that this is necessary; it is to protect you from theft.

F14 Practice 3 Problem 2
Linear Algebra, Dave Bayer

[2] Find the determinant of the matrix

$$A = \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 \\ 1 & 3 \end{matrix}} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{matrix} & \boxed{\begin{matrix} 3 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{matrix}} \end{bmatrix}$$

det(A) = $\boxed{60}$

matrix is block triangular

$$\Rightarrow \det(A) = \underbrace{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}}_{5} \cdot \underbrace{\begin{vmatrix} 3 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 1 & 2 \end{vmatrix}}_{-1\underbrace{\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}}_{0} + 3\underbrace{\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}}_{4}} = 5 \cdot 12 = 60$$

check: subtract row 5 from row 3:

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 3 & 0 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix} \qquad \underbrace{\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}}_{5} \cdot 2 \cdot 3 \cdot 2 = 60 \quad ☑$$

F14 Practice 3 Problem 3
Linear Algebra, Dave Bayer

[3] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

check final answer against original matrix. $AA^{-1} = I$ ✓

$$A^{-1} = \frac{1}{\boxed{7}} \begin{bmatrix} \boxed{1} & \boxed{2} & \boxed{4} \\ \boxed{1} & \boxed{2} & \boxed{-3} \\ \boxed{3} & \boxed{-1} & \boxed{-2} \end{bmatrix}$$

$$\begin{array}{c|cccc|} 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 2 & -1 & 0 & 2 \\ 2 & -1 & 0 & 2 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 & 2 \\ \hline \end{array}$$

$$\underbrace{\begin{bmatrix} +1 & +2 & +4 \\ +1 & +2 & -3 \\ +3 & -1 & -2 \end{bmatrix} \Big/ 7 \begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}}_{\text{check ✓ } A^{-1}A = I} = I$$

- transpose
- copy first two cols
- copy first two rows
- box all but first row, col
- mark 9 dots
- take minors around 9 dots
- figure out denominator and adjust signs if needed.
- check final answer against original matrix, to catch any copying errors.

F14 Practice 3 Problem 4
Linear Algebra, Dave Bayer

[4] Using Cramer's rule, solve for x in the system of equations

$$\begin{bmatrix} 3 & 1 & a \\ 2 & 1 & b \\ 1 & 1 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

$$x = \frac{(\boxed{3})\,a + (\boxed{-3})\,b + (\boxed{0})\,c}{(\boxed{1})\,a + (\boxed{-2})\,b + (\boxed{1})\,c}$$

$$x = \frac{\begin{vmatrix} 5 & 1 & a \\ 5 & 1 & b \\ 2 & 1 & c \end{vmatrix}}{\begin{vmatrix} 3 & 1 & a \\ 2 & 1 & b \\ 1 & 1 & c \end{vmatrix}} = \frac{a\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} - b\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} + c\begin{vmatrix} 5 & 1 \\ 5 & 1 \end{vmatrix}}{a\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - b\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + c\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix}}$$

$$= \boxed{\frac{3a - 3b + 0c}{a - 2b + c}}$$

check:

$$\begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 2 \end{bmatrix}$$

guess sample values that work together

$a=1,\ b=2,\ c=0 \Rightarrow x=1$

$$x = \frac{3\cdot 1 - 3\cdot 2 + 0\cdot 0}{1 - 2\cdot 2 + 0} = \frac{-3}{-3} = 1 \checkmark$$

F14 Practice 3 Problem 5
Linear Algebra, Dave Bayer

[5] Find the characteristic equation and a system of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\lambda^2 + (\boxed{-4})\lambda + (\boxed{3}) = 0$$

$$\lambda_1,\ \lambda_2 = \boxed{1},\ \boxed{3}$$

$$v_1,\ v_2 = \begin{bmatrix} \boxed{1} \\ \boxed{-1} \end{bmatrix},\ \begin{bmatrix} \boxed{1} \\ \boxed{1} \end{bmatrix}$$

$$\lambda^2 - \underset{(2+2)}{\text{trace}(A)}\lambda + \underset{(2\cdot 2 - 1\cdot 1)}{\det(A)} = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1,\ 3$$

$$\lambda = 1 \quad A - \lambda I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\lambda = 3 \quad A - \lambda I: \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

---

check

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

F14 Practice 3 Problem 6
Linear Algebra, Dave Bayer

[6] Find the characteristic equation and a system of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\lambda^3 + (-5)\,\lambda^2 + (6)\,\lambda + (0) = 0$$

$$\lambda_1,\ \lambda_2,\ \lambda_3 = 0,\ 2,\ 3$$

$$v_1,\ v_2,\ v_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix},\ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$\lambda^3 - \text{trace}(A)\lambda^2 + \text{huh}(A)\lambda - \det(A) = 0$$

$$\text{trace}(A) = 2+1+2 = 5$$

$$\underbrace{\text{huh}}_{\text{(not really its name)}}(A) = \begin{vmatrix} 2 & 2 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} = -2+4+4 = 6$$

$$\det(A) = 2\begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} = 0$$

$$\lambda^3 - 5\lambda^2 + 6\lambda = 0$$

$$\lambda(\lambda-2)(\lambda-3) = 0$$

$$\lambda = 0, 2, 3$$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0 \qquad \lambda = 0$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & -1 & 1 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 0 \qquad \lambda = 2$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & -2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = 0 \qquad \lambda = 3$$

---

check:

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 6 \\ 0 & 0 & 3 \\ 0 & -4 & -6 \end{bmatrix} \quad ☑$$

$$\lambda = 0 \quad \lambda = 2 \quad \lambda = 3$$

F14 Practice 3 Problem 7
Linear Algebra, Dave Bayer

[7] Express $f(n)$ using a matrix power, and find $f(8)$, where

$$f(0) = 1, \quad f(1) = 2$$
$$f(n) \;=\; 2\,f(n-1) \;-\; f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}^{A}{}^{n} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \qquad \det = 1$$

$$f(8) \;=\; 9$$

check as we go

$$A^2 = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad \det = 1 = 1^2 \; ☑$$

$$A^4 = \begin{bmatrix} -3 & 4 \\ -4 & 5 \end{bmatrix} \quad \det = 1 \; ☑$$

$$A^8 = \begin{bmatrix} -7 & 8 \\ -8 & 9 \end{bmatrix} \quad \det = 1 \; ☑ \qquad \begin{bmatrix} f(8) \\ \end{bmatrix} = \begin{bmatrix} -7 & 8 \\ & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 9$$

---

pattern guess $A^n = \begin{bmatrix} 1-n & n \\ -n & n+1 \end{bmatrix}$ $\qquad \begin{bmatrix} f(n) \\ \end{bmatrix} = \begin{bmatrix} 1-n & n \\ & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2n+1-n = n+1$? ☑

check:

n	$f(n)$
0	1
1	2
2	3
3	4
4	5
5	6
6	7
7	8
8	9 ☑

$3 = 2\cdot 2 - 1$

$4 = 2\cdot 3 - 2$

$\vdots$

F14 Practice 3 Problem 8
Linear Algebra, Dave Bayer

[8] Express $f(n)$ using a matrix power, and find $f(8)$, where

$$f(0)=1, \quad f(1)=1, \quad g(1)=2$$
$$f(n) \;=\; f(n-1) \;+\; g(n-1) \;+\; f(n-2)$$
$$g(n) \;=\; f(n-1) \;-\; g(n-1) \;+\; f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \\ g(n+1) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}}^{A}{}^{n} \begin{bmatrix} f(0) \\ f(1) \\ g(1) \end{bmatrix}$$

$$f(8) \;=\; 202$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix} \qquad A^4 = \begin{bmatrix} 3 & 5 & 3 \\ 8 & 11 & 2 \\ 2 & 5 & 4 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 55 & 85 & 31 \\ * & * & * \\ * & * & * \end{bmatrix}$$

$$\begin{array}{c|cccl} 3 & 3 & 5 & 3 & = \;\; 9 \;\; 15 \;\; 9 \\ 5 & 8 & 11 & 2 & = \;\; 40 \;\; 55 \;\; 10 \\ 3 & 2 & 5 & 4 & = \;\; 6 \;\; 15 \;\; 12 \\ \hline & & & & \;\;\;\;\; 55 \;\; 85 \;\; 31 \end{array}$$

$$\begin{bmatrix} f(8) \\ * \\ * \end{bmatrix} = \begin{bmatrix} 55 & 85 & 31 \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 202 \\ * \\ * \end{bmatrix}$$

$$\underbrace{55+85}_{140}+62 = 202$$

check:

n	f(n)	g(n)
0	1	
1	1	2
2	4	0
3	5	5
4	14	4
5	23	15
6	52	22
7	97	53
8	202	

$\boxed{\checkmark}$

f: $\begin{array}{ll} + & \\ + & + \end{array}$ g: $\begin{array}{ll} + & \\ + & - \end{array}$

$\begin{array}{ll} 1 & \\ 1 & 2 \\ 4 & \end{array}$ $\qquad \begin{array}{ll} 1 & \\ 1 & 2 \\ 0 & \end{array}$

learn recursion as visual pattern that we slide down table as we calculate

F14 Practice 3 Problem 9
Linear Algebra, Dave Bayer

[9] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

expand by minors

$1 \qquad 1 \qquad -1 \qquad 1(-1)-2\cdot 1=-3$

$$[\,] \quad \begin{bmatrix}1\end{bmatrix} \quad \begin{bmatrix}1&2\\1&1\end{bmatrix} \quad \begin{bmatrix}1&2&0\\1&1&2\\0&1&1\end{bmatrix} \quad \begin{bmatrix}1&2&0&0\\1&1&2&0\\0&1&1&2\\0&0&1&1\end{bmatrix} \quad \begin{bmatrix}1&2&0&0&0\\1&1&2&0&0\\0&1&1&2&0\\0&0&1&1&2\\0&0&0&1&1\end{bmatrix}$$

Find $f(0)$ and $f(1)$. Find a recurrence relation for $f(n)$. Express $f(n)$ using a matrix power. Find $f(8)$.

$$f(0) = \boxed{1} \qquad f(1) = \boxed{1}$$

$$f(n) = \left(\boxed{1}\right) f(n-1) + \left(\boxed{-2}\right) f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{1} \\ \boxed{-2} & \boxed{1} \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$f(8) = \boxed{-17}$$

$$f(n) = +1 \underbrace{\left|\ \blacksquare\ \right|}_{f(n-1)} - 2 \underbrace{\left| \begin{matrix} 1 & \cdots \\ 0 & \blacksquare \\ 0 & \end{matrix} \right|}_{f(n-2)}$$

$$f(n) = f(n-1) - 2f(n-2)$$

n	f(n)
0	1
1	1
2	-1
3	-3
4	-1
5	5
6	7
7	-3
8	-17

$\boxed{\begin{matrix}-2\\+1\end{matrix}}$ sliding pattern

$$A^2 = \begin{bmatrix} -2 & 1 \\ -2 & -1 \end{bmatrix} \quad \det = 4 = 2^2 \ ☑$$

$$A^4 = \begin{bmatrix} 2 & -3 \\ 6 & -1 \end{bmatrix} \quad \det = 16 \ ☑$$

$$A^8 = \begin{bmatrix} -14 & -3 \\ \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} f(8) \\ \cdot \end{bmatrix} = \begin{bmatrix} -14 & -3 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -17 \ ☑$$

check $\leftrightarrow$