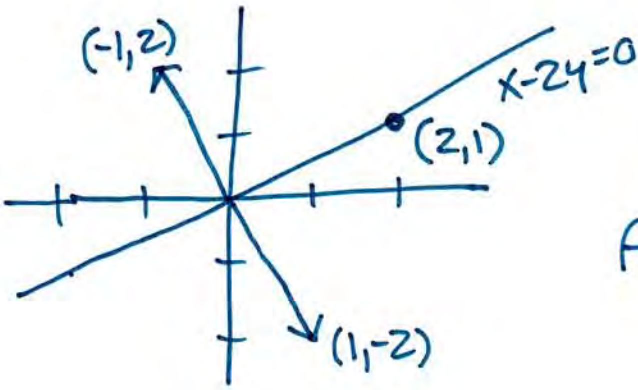
F14 Practice 2 Problem 1
Linear Algebra, Dave Bayer

[1] Find the 2×2 matrix which reflects across the line $x - 2y = 0$.



$(2,1) \mapsto (2,1)$ fixed
$(1,-2) \mapsto (-1,2)$ reflected

$$A \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} +2 & +1 \\ +1 & -2 \end{bmatrix} \Big/ {+5} = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \Big/ 5$$

check: $$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \Big/ 5 \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ 5 & 10 \end{bmatrix} \Big/ 5 = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \Big/ 5}$$

(please box your answer so I can find it when grading.)

F14 Practice 2 Problem 2
Linear Algebra, Dave Bayer

[2] Find the 3×3 matrix which vanishes on the plane $x + y + z = 0$, and maps the vector $(1, 0, 0)$ to itself.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

enforce $x+y+z=0$

make image all multiples of $(1,0,0)$

test: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ no rescaling needed

$$\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

in general, the rank one matrix

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} d & e & f \end{bmatrix} \Big/ (a,b,c)\cdot(d,e,f) = \begin{bmatrix} ad & ae & af \\ bd & be & bf \\ cd & ce & cf \end{bmatrix} \Big/ (a,b,c)\cdot(d,e,f)$$

vanishes on $dx+ey+fz=0$, and sends (a,b,c) to itself.

check: $$\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} d & e & f \end{bmatrix} \right) \Big/ (a,b,c)\cdot(d,e,f) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \frac{(d,e,f)\cdot(a,b,c)}{(a,b,c)\cdot(d,e,f)} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} ✓$$

F14 Practice 2 Problem 3
Linear Algebra, Dave Bayer

[3] Find the 3×3 matrix which $\underbrace{\text{vanishes on the vector } (1,1,1), \text{ and maps each point on the plane } x+y=0 \text{ to itself}}_{A}$.

B maps $(1,1,1)$ to itself, vanishes on $x+y=0$

$$A+B=I \implies B=I-A$$
$$\boxed{A=I-B}$$

$$B = \begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\end{bmatrix}/2 = \begin{bmatrix}1&1&0\\1&1&0\\1&1&0\end{bmatrix}/2$$

$$A = I-B = \begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}/2 - \begin{bmatrix}1&1&0\\1&1&0\\1&1&0\end{bmatrix}/2 = \boxed{\begin{bmatrix}1&-1&0\\-1&1&0\\-1&-1&2\end{bmatrix}/2}$$

check: $$\begin{bmatrix}1&-1&0\\-1&1&0\\-1&-1&2\end{bmatrix}/2 \begin{bmatrix}1&1&0\\1&-1&0\\1&\underbrace{0}_{\text{in plane } x+y=0}&1\end{bmatrix} = \begin{bmatrix}0&2&0\\0&-2&0\\0&0&2\end{bmatrix}/2 = \begin{bmatrix}0&1&0\\0&-1&0\\0&0&1\end{bmatrix}$$ ✓

F14 Practice 2 Problem 4
Linear Algebra, Dave Bayer

[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} t$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} / 6 = \boxed{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} / 6}$$

check: $$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} / 6 \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 2 & \underbrace{0 \;\; -1}_{\perp \text{ to line}} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 6 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix} / 6 \quad \boxed{\checkmark}$$

F14 Practice 2 Problem 5
Linear Algebra, Dave Bayer

[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + y + z = 0$$

$$I - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} / 3 = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix} / 3 - \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} / 3$$

$$= \boxed{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} / 3}$$

check: $$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} / 3 \; \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}}_{\text{vectors in plane}} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix} / 3 \quad \checkmark$$

F14 Practice 2 Problem 6
Linear Algebra, Dave Bayer

[6] Find the row space and the column space of the matrix redundant columns

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 \end{bmatrix}$$

independent columns

so rank is $\underline{\underline{2}}$. Need two columns for columnspace
Need two rows for rowspace

basis for column space: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

basis for row space: $\left\{ \begin{matrix} [1\ 1\ 1\ 1\ 1], \\ [1\ 2\ 2\ 2\ 2] \end{matrix} \right\}$

F14 Practice 2 Problem 7
Linear Algebra, Dave Bayer

[7] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Downarrow \qquad (Ax = b \Rightarrow A^TAx = A^Tb)$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

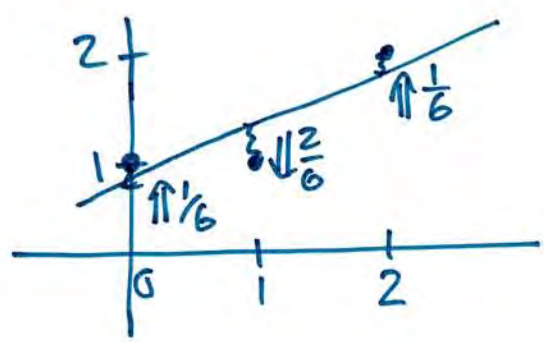
$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Big/ 6$$

$$\boxed{y = \tfrac{1}{2}x + 5/6}$$

check

x_i	y_i	$\frac{1}{2}x + 5/6$	Δ
0	1	5/6	-1/6
1	1	8/6	2/6
2	2	11/6	-1/6



↳ $(-1,2,-1)$ is $\perp$ to columns of A

$\Leftrightarrow$ "springs balance" in stick drawing above

F14 Practice 2 Problem 8
Linear Algebra, Dave Bayer

[8] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,-1,0,0) \qquad (0,1,-1,0) \qquad (0,0,1,-1) \qquad (1,0,-1,0) \qquad (0,1,0,-1)$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

(One can use Gram-Schmidt, as shown in class.)

Here, all vectors satisfy $w+x+y+z=0$
and 3 start in different positions $\Rightarrow$ are independent

So find $\perp$ basis for $w+x+y+z=0$:

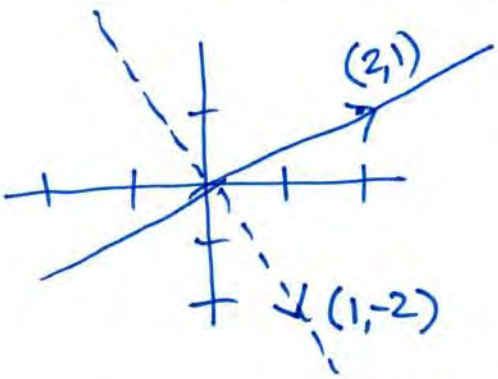
$$\mathbb{R}^4 \left\{ \begin{array}{l} V \left\{ \begin{array}{l} (1,-1,0,0) \\ (1,1,-2,0) \quad \leftarrow \perp \text{ to first, and sum to } 0 \\ (1,1,1,-3) \quad \leftarrow \perp \text{ to previous two, and sum to } 0 \end{array} \right. \\ (1,1,1,1) \quad \leftarrow (1,1,1,1)\cdot(w,x,y,z)=0 \end{array} \right.$$

is equation defining V

$\Leftrightarrow$ $(1,1,1,1)$ is $\perp$ to V

F14 Practice 2 Problem 1
Linear Algebra, Dave Bayer

[1] Find the 2×2 matrix which reflects across the line $x - 2y = 0$.



(2,1) on line

$(1,-2)\cdot(2,1) = 0$

$\Rightarrow (1,-2) \perp$ line.

$$\left[\begin{array}{l} (2,1) \longmapsto (2,1) \\ (1,-2) \longmapsto -(1,-2) = (-1,2) \end{array}\right]$$

To determine A, find images of basis (1,0) and (0,1)

$$(1,0) = \frac{2(2,1)+(1,-2)}{5}$$ } get 2^{nd} entry to cancel to 0 then scale by 5 to make first entry = 1

$$\longmapsto \frac{2(2,1)-(1,-2)}{5} = (3,4)/5$$

$$(0,1) = \frac{(2,1)-2(1,-2)}{5}$$ } make first entry 0 then scale so second entry 1

$$\longmapsto \frac{(2,1)+2(1,-2)}{5} = (4,-3)/5$$

so $$\boxed{A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} / 5}$$

check:

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} / 5 \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -5 \\ 5 & 10 \end{bmatrix} / 5 = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

✓

F14 Practice 2 Problem 2
Linear Algebra, Dave Bayer

sends to zero

[2] Find the 3×3 matrix which vanishes on the plane $x + y + z = 0$, and maps the vector $(1, 0, 0)$ to itself.

Write each basis vector $(1,0,0), (0,1,0), (0,0,1)$
as a sum of a multiple of $(1,0,0)$ and a vector in plane.

$\circledast$ $(1,0,0) = (1,0,0) + (0,0,0) \longmapsto (1,0,0)$

$\circledast$ $(0,1,0) = r(1,0,0) + v$ for v in plane

$v = (-r,1,0)$ $\quad (1,1,1)\cdot(-r,1,0) = 0$

$1 - r = 0 \quad r = 1$

$\Rightarrow (0,1,0) = (1,0,0) + (-1,1,0)$

$\longmapsto (1,0,0) + 0 = (1,0,0)$

$\circledast$ $(0,0,1) = (1,0,0) + (-1,0,1)$ (same pattern as before)

$\longmapsto (1,0,0) + 0 = (1,0,0)$

check:

$$\Rightarrow \boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

fixes $(1,0,0)$ ✓

vanishes when
$(1,1,1)\cdot(x,y,z) = 0$ ✓

F14 Practice 2 Problem 3
Linear Algebra, Dave Bayer

[3] Find the 3×3 matrix which vanishes on the vector $(1,1,1)$, and maps each point on the plane $x+y=0$ to itself.

Write basis vectors in terms of $(1,1,1)$ and plane

$$\circledast \quad \underbrace{(1,0,0)}_{} = \tfrac{1}{2}\underbrace{(1,1,1)}_{} + \left(\tfrac{1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}\right)$$

x+y same on both sides, so difference will be zero

$$\circledast \quad (0,1,0) = \tfrac{1}{2}(1,1,0) + \left(\tfrac{-1}{2}, \tfrac{1}{2}, \tfrac{-1}{2}\right)$$ by symmetry (swap x and y)

$$\circledast \quad (0,0,1) = (0,0,0) + (0,0,1)$$

$$\Rightarrow \boxed{A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} / 2}$$

$\boxed{\checkmark}$

check:
$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} / 2 \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} / 2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in plane

to zero

fixed

F14 Practice 2 Problem 4
Linear Algebra, Dave Bayer

[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} t$$

write each basis vector as multiple of $(1,1,2)$ and $\perp$ to $(1,1,2)$

$\circledast$ $(1,0,0) = t(1,1,2) + v$ where $v \cdot (1,1,2) = 0$

$$v = (1,0,0) - t(1,1,2)$$

$$0 = (1,1,2)\cdot v = (1,1,2)\cdot(1,0,0) - t(1,1,2)\cdot(1,1,2)$$
$$= 1 - 6t \Rightarrow t = 1/6$$

$$(1,0,0) = (1,1,2)/6 + (5,-1,-2)/6 \quad ✓$$

$\circledast$ $(0,1,0) = (1,1,2)/6 + (-1,5,-2)/6$ by symmetry (swap x and y)

$\circledast$ $(0,0,1) = (1,1,2)/3 + (-1,-1,1)/3$ ✓

it felt like this had to be a 3 this time, and it checks

So
$$(1,0,0) \mapsto (1,1,2)/6$$
$$(0,1,0) \mapsto (1,1,2)/6$$
$$(0,0,1) \mapsto (2,2,4)/6$$

$$\boxed{A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} /6}$$

check:
$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} /6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & \underbrace{0 \;\; -1}_{\text{in plane}} \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 6 & 0 & 0 \\ 12 & 0 & 0 \end{bmatrix} /6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} ✓$$

F14 Practice 2 Problem 5
Linear Algebra, Dave Bayer

[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + y + z = 0$$

Write each basis vector as a multiple of (1,1,1) and a vector in plane

$$\circledast \quad \underbrace{(1,0,0)} = \underbrace{(1,1,1)}/3 + (2,-1,-1)/3$$

same sum, so difference sums to zero, ⇒ in plane

$$\circledast \quad (0,1,0) = (1,1,1)/3 + (-1,2,-1)/3$$
$$\circledast \quad (0,0,1) = (1,1,1)/3 + (-1,-1,2)/3$$
by symmetry

$$\Rightarrow \boxed{A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} /3}$$

so
$$(1,0,0) \mapsto (2,-1,-1)/3$$
$$(0,1,0) \mapsto (-1,2,-1)/3$$
$$(0,0,1) \mapsto (-1,-1,2)/3$$

check:
$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} /3 \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & \underbrace{0}_{\text{in plane}} & -1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix} /3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \checkmark$$

F14 Practice 2 Problem 6
Linear Algebra, Dave Bayer

[6] Find the row space and the column space of the matrix

$$\begin{array}{cc} & 2 \\ 2 & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 2 & 3 & 3 & 3 & 3 \end{bmatrix} \end{array}$$

row 1, row 2 are not multiples of each other
row 3 = row 1 + row 2 $\Rightarrow$ dependent
$\Rightarrow$ rank = 2, and row 1, row 2 give basis for row space

$$\boxed{\text{row space basis} = \left\{ \begin{array}{c} (1,1,1,1,1) \\ (1,2,2,2,2) \end{array} \right\}}$$

column space has <u>same</u> rank $\Rightarrow$ dim 2 also
need two independent columns.
col 1, col 2 are not multiples

$$\boxed{\text{col space basis} = \left\{ \begin{array}{c} (1,1,2) \\ (1,2,3) \end{array} \right\}}$$

check: other columns are dependent on these columns. ✓
" rows " " " " rows. ✓

F14 Practice 2 Problem 7
Linear Algebra, Dave Bayer

[7] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

2x2 formula:

$$\begin{bmatrix} \Sigma x_i^2 & \Sigma x_i \\ \Sigma x_i & \Sigma 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \Sigma x_i y_i \\ \Sigma y_i \end{bmatrix}$$

(A 1970's calculator needed just five registers, for $\Sigma 1, \Sigma x, \Sigma x^2, \Sigma y, \Sigma xy$ to calculate this line for arbitrarily many points)

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

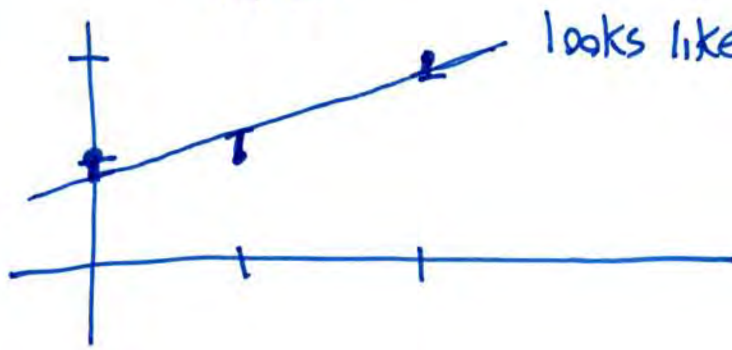
$$= \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \Big/ 6$$

$a = 1/2 \quad b = 5/6$

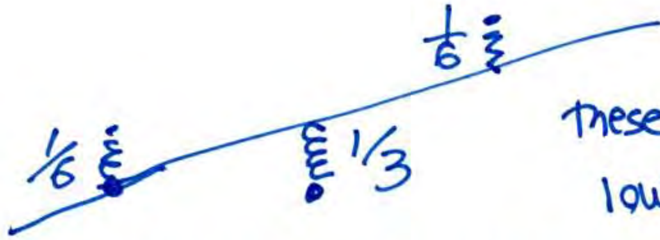
$$\boxed{y = \tfrac{1}{2}x + 5/6}$$

x	y	$\frac{1}{2}x + 5/6$	Δ
0	1	5/6	−1/6
1	1	8/6	1/3
2	2	11/6	−1/6

check:



looks like good fit



these springs won't raise lower or twist line ☑

or instead $(-1, 2, -1)/6 \cdot (1,1,1) = 0$ ☑
$\cdot (0,1,2) = 0$ ☑ (really, same idea as springs)

F14 Practice 2 Problem 8
Linear Algebra, Dave Bayer

[8] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,-1,0,0) \qquad (0,1,-1,0) \qquad (0,0,1,-1) \qquad \underbrace{(1,0,-1,0) \qquad (0,1,0,-1)}_{\text{dependent on previous}}$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

Gram Schmidt

$v_1 = 1\ {-1}\ 0\ 0$
$v_2 = 0\ 1\ {-1}\ 0$
$v_3 = 0\ 0\ 1\ {-1}$
$v_4 = 0\ 0\ 0\ 1$

v_4 is any vector not in V

$$w_1 = v_1 = \boxed{(1,-1,0,0)}$$

$$w_2 = v_2 - \frac{v_2\cdot w_1}{w_1\cdot w_1}\, w_1$$
$$= (0,1,-1,0) - \frac{(-1)}{2}(1,-1,0,0)$$
$$= (0,1,-1,0) + \tfrac{1}{2}(1,-1,0,0)$$
$$= (1,1,-2,0)/2$$
$$\to \boxed{(1,1,-2,0)} \text{ (rescaled)}$$

check as we go
☑ in V
☑ $\perp$ to previous

$$w_3 = v_3 - \left(\frac{v_3\cdot w_1}{w_1\cdot w_1}\right)w_1 - \left(\frac{v_3\cdot w_2}{w_2\cdot w_2}\right)w_2$$
$$= (0,0,1,-1) - \left(\frac{0}{2}\right)w_1 - \left(\frac{-2}{6}\right)(1,1,-2,0)$$
$$= (1,1,1,-3)/3 \quad \text{rescale to } \boxed{(1,1,1,-3)}$$

$$w_4 = (0,0,0,1) - \frac{0}{2}w_1 - \frac{0}{2}w_2 - \left(\frac{-3}{12}\right)(1,1,1,-3) = (+1,+1,+1,+1)/4$$

rescale to $(1,1,1,1)$
☑ $\perp$ to rest

$$\boxed{\text{all of } \mathbb{R}^4 \begin{cases} V \begin{cases} (1,-1,0,0) \\ (1,1,-2,0) \\ (1,1,1,-3) \end{cases} \\ (1,1,1,1) \end{cases}}$$

[1] Find the 2×2 matrix which reflects across the line $x - 2y = 0$.

$(2,1)$ on line
$(1,-2)$ $\perp$ to line

Let $S = \{(1,0),(0,1)\}$ usual (standard) coordinates
$V = \{(2,1),(1,-2)\}$ good coords for seeing reflection

$L(2,1) = (2,1)$ $\quad v_1 \mapsto v_1$
$L(1,-2) = (-1,+2)$ $\quad v_2 \mapsto -v_2$ $\quad$ so $\underset{V \leftarrow V}{L} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

translate to standard S coordinates:

$$\underset{S \leftarrow S}{\overset{L}{\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}/5}} = \underset{S \leftarrow V}{\overset{\text{translate}}{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}}} \underbrace{\underset{V \leftarrow V}{\overset{L}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}} \underset{V \leftarrow S}{\overset{\text{translate}}{\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}/5}}}_{\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}/5}$$

$\leftarrow$ inverse of $S \leftarrow V$ translation

check:

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}/5 \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \checkmark$$

F14 Practice 2 Problem 2
Linear Algebra, Dave Bayer

[2] Find the 3×3 matrix which vanishes on the plane $x + y + z = 0$, and maps the vector $(1, 0, 0)$ to itself.

$$\left.\begin{array}{l} v_1 = (1,0,0) \\ v_2 = (1,-1,0) \\ v_3 = (0,1,-1) \end{array}\right\} \text{ V coords for seeing map clearly}$$

$$v_1 \mapsto v_1$$
$$v_2, v_3 \mapsto 0$$

$$\underset{S \leftarrow S}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} = \underset{S \leftarrow V}{\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}} \underset{V \leftarrow V}{\begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}} \underbrace{\underset{V \leftarrow S}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}}}_{\text{inverse of } S \leftarrow V}$$

find inverse by guessing triangular shape, filling in from diagonal up so product is identity:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \boxed{} & \boxed{} \\ 0 & -1 & \boxed{} \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \boxed{} \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} = I$$

$$A = \begin{bmatrix} 1 & & \\ 0 & & \\ 0 & & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

doesn't matter, not used
might as well drop these columns and corresponding rows

F14 Practice 2 Problem 3
Linear Algebra, Dave Bayer

[3] Find the 3×3 matrix which vanishes on the vector $(1, 1, 1)$, and maps each point on the plane $x + y = 0$ to itself.

$$V \left\{ \begin{array}{l} (1,1,1) \mapsto 0 \\ (1,-1,0) \mapsto (1,-1,0) \\ (0,0,1) \mapsto (0,0,1) \end{array} \right.$$

find inverse by row reduction

$$\underset{S \leftarrow S}{\begin{bmatrix} A \end{bmatrix}} = \underset{S \leftarrow V}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{V \leftarrow V}{\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \underset{V \leftarrow S}{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & & \\ 1 & -1 & 0 & & 1 & \\ 1 & 0 & 1 & & & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & -2 & 0 & 2 \end{array}\right] \Big/ 2$$

② ← ② − ①, ③ ← ③ − ① ② ← $-\frac{1}{2}$②, rescale

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right] \Big/ 2$$

① ← ① − ②, ③ ← ③ + ②

check: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Big/ 2$ ✓

so $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 2 = \boxed{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 2}$

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or guess inverse using block triangular form

$$\left[\begin{array}{c|c} B & 0 \\ \hline /// & C \end{array}\right] \left[\begin{array}{c|c} B^{-1} & 0 \\ \hline /// & C^{-1} \end{array}\right] = \left[\begin{array}{cc|c} 1 & & \\ & 1 & \\ \hline & & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Big/ 2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \\ \hline ? & & 2 \end{array}\right] \Big/ 2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 2}_{\text{inverse}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

F14 Practice 2 Problem 4
Linear Algebra, Dave Bayer

[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} t$$

$$A = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}}_{\perp \text{ to line}} \underbrace{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}}_{V \leftarrow V} \underbrace{\begin{bmatrix} & & \\ & & \\ & \boxed{} & \end{bmatrix}}_{V \leftarrow S}$$

only need this row of inverse, so figure it out

$$\begin{bmatrix} & & \\ & & \\ & \boxed{} & \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ 0 & 0 & 1 \end{bmatrix}$$

(we swap $A^{-1} \cdot A = I$ to isolate row)

$\perp$ to $(1,-1,0) \Rightarrow (1,1,\ )$ whatever we need

$\perp$ to $(1,1,-1) \Rightarrow (1,1,2)$ or multiple

$\cdot\, (1,1,2) = 1 \Rightarrow (1,1,2)/6$

$$A = \begin{bmatrix} & & 1 \\ & & 1 \\ & & 2 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & & \\ & & \\ 1 & 1 & 2 \end{bmatrix} / 6$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \begin{pmatrix} 1 & 1 & 2 \end{pmatrix} / 6 = \boxed{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} / 6}$$

F14 Practice 2 Problem 5
Linear Algebra, Dave Bayer

[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + y + z = 0$$

$$A = \underset{S \leftarrow V}{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}} \underset{V \leftarrow V}{\begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}} \underset{V \leftarrow S}{\begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & -2 \\ +2 & -1 & -1 \end{bmatrix}} \Big/ (+3)$$

use formula for 3x3 inverse:

$$\begin{array}{c|cccc} 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \end{array}$$

(described on our course materials page)

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{bmatrix} \Big/ 3 = \boxed{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \Big/ 3}$$

1)

F14 Practice 2 Problem 9
Linear Algebra, Dave Bayer

[9] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(1) = 0$. Find the orthogonal projection of the polynomial [redacted] onto the subspace W, with respect to the inner product

$X+1$

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

$V = \{ax^2+bx+c\} \cong \{(a,b,c)\} = \mathbb{R}^3$

$W: \; f(x) = ax^2+bx+c$
$\quad f(1) = a+b+c = 0$ one condition $W \cong \mathbb{R}^2$

Find $\perp$ basis for W. $x-1$ and ??

$\langle ax^2+bx+c, x-1\rangle = \int_0^1 (ax^3+(b-a)x^2+(c-b)x+(-c))\,dx$
$= \frac{1}{4}a + \frac{1}{3}(b-a) + \frac{1}{2}(c-b) + -c = 0$

$\times 12 \qquad 3a+4(b-a)+6(c-b)-12c=0$

to be $\perp$ to $x-1$ $\Longrightarrow$ $+a+2b+6c=0$
to be in W $\Longrightarrow$ $a+b+c=0$ subtract

$b+5c=0$
$b=5 \; c=-1 \Rightarrow a=-4$

(1) $\boxed{\text{So } x-1,\; 4x^2-5x+1 \text{ is } \perp \text{ basis for W.}}$

check $x-1\big|_{x=1}=0$, $4x^2-5x+1\big|_{x=1}=0$
So both are in W.

$\langle x-1, 4x^2-5x+1\rangle = \int_0^1 (4x^3-9x^2+6x-1)\,dx$
$= 1 - \frac{9}{3} + \frac{6}{2} - 1 = 0$ ✓

$$\begin{array}{c|ccc} & 4x^2 & -5x & 1 \\ \hline x & 4 & -5 & 1 \\ -1 & -4 & 5 & -1 \\ \hline & -9 & 6 & \end{array}$$

Now compute projection of $x+1$ onto each vector, and add. (Gram-Schmidt).

$\langle x+1, 4x^2-5x+1\rangle = \int_0^1 (4x^3-x^2-4x+1)\,dx$
$= \frac{4}{4} - \frac{1}{3} - 4/2 + 1 = ??$

$$\begin{array}{c|ccc} & 4x^2 & -5x & 1 \\ \hline x & 4 & -5 & 1 \\ 1 & 4 & -5 & 1 \end{array}$$

Easier to use $60\langle f,g\rangle$ because no denominators, and the 60s will cancel out

2)

$$60\langle x+1, x-1\rangle = 60\int_0^1 (x^2-1)\,dx = 20-60 = -40.$$

$$\boxed{\begin{array}{ccccc} x^4 & x^3 & x^2 & x & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 12 & 15 & 20 & 30 & 60 \end{array}}$$

$$60\langle x-1, x-1\rangle = \begin{array}{ccc} x^2 & -2x & +1 \\ \downarrow & \downarrow & \downarrow \\ 20 & -2\cdot 30 & +60 \end{array} = 20$$

so $\text{proj } x+1 \text{ onto } x-1 = \dfrac{\langle x+1, x-1\rangle}{\langle x-1, x-1\rangle}(x-1) = \dfrac{-40}{20}(x-1)$

$$= \boxed{-2x+2}$$

$$60\langle x+1, 4x^2-5x+1\rangle = \begin{array}{cccc} 4x^3 & -x^2 & -4x & +1 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4\cdot 15 & -20 & -4\cdot 30 & +60 \end{array} = -20$$

$$\begin{array}{c|ccc} & 4 & -5 & 1 \\ \hline 1 & 4 & -5 & 1 \\ 1 & 4 & -5 & 1 \\ \hline 4 & -1 & -4 & 1 \end{array}$$

$$60\langle 4x^2-5x+1, 4x^2-5x+1\rangle = \begin{array}{ccccc} 16x^4 & -40x^3 & +33x^2 & -10x & +1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 16\cdot 12 & -40\cdot 15 & +33\cdot 20 & -10\cdot 30 & +60 \end{array}$$

$$\begin{array}{c|ccc} & 4 & -5 & 1 \\ \hline 4 & 16 & -20 & 4 \\ -5 & -20 & 25 & -5 \\ 1 & 4 & -5 & 1 \\ \hline \end{array}$$
$$16 \quad -40 \quad 33 \quad -10 \quad 1$$

$$\begin{array}{r} 192 \\ 660 \\ 60 \\ \hline 912 \end{array} - \begin{array}{r} 600 \\ 300 \\ \hline 900 \end{array} = 12$$

so $\text{proj } x+1 \text{ onto } 4x^2-5x+1 = \dfrac{\langle x+1, 4x^2-5x+1\rangle}{\langle 4x^2+5x-1, 4x^2-5x+1\rangle}(4x^2-5x+1)$

$$= \frac{-20}{12}(4x^2-5x+1) = \frac{-5}{3}(4x^2-5x+1) = \boxed{-\frac{20}{3}x^2 + \frac{25}{3}x - \frac{5}{3}}$$

$$-\frac{6}{3}x + \frac{6}{3}$$

projection of $x+1$ into W $\Longrightarrow$ $\boxed{\boxed{-\frac{20}{3}x^2 + \frac{19}{3}x + \frac{1}{3}}}$

---

check: Is difference $\perp$ to W?

subtract: $-\frac{3}{3}x - \frac{3}{3}$

$$-\frac{20}{3}x^2 + \frac{16}{3}x - \frac{2}{3}$$

rescale to $10x^2 + 8x + 1$

3) check, continued.

Easier basis for W is $x-1,\ x^2-1$

Is $10x^2-8x+1 \perp$ to both of these?

$$60\langle 10x^2-8x+1, x-1\rangle = 10x^3-18x^2+9x-1$$

$$\begin{array}{r|rrr} & 10 & -8 & 1 \\ \hline 1 & 10 & -8 & 1 \\ -1 & -10 & 8 & -1 \end{array}$$

$$10\cdot15-18\cdot20+9\cdot30-60$$

$$150-360+270-60=0 \ \checkmark$$

$$60\langle 10x^2-8x+1, x^2-x\rangle = 10x^4-18x^3+9x^2-x$$

(same shifted by x)

$$10\cdot12-18\cdot15+9\cdot20-30$$

$$120-270+180-30=0 \ \checkmark$$

So projection $x+1$ onto W is

$$\boxed{-\frac{20}{3}x^2+\frac{19}{3}x+\frac{1}{3}}$$

4/

F14 Practice 2 Problem 9
Linear Algebra, Dave Bayer

[9] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(1) = 0$. Find the orthogonal projection of the polynomial ~~x~~ $x+1$ onto the subspace W, with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

There must be an easier way!!

Want to write $x+1 = \underbrace{(\sim\sim\sim)}_{\perp \text{ to } W} + \underbrace{(\sim\sim)}_{\text{in } W}$

in which case second part is projection.
Easy to test for "in W", plug in 1 and see if $f(1)=0$.

W is a plane, so a unique direction is $\perp$ to W. Find it.
$x-1$, x^2-x is nice basis for W, solve for a poly $\perp$ to these.

$$60\langle ax^2+bx+c, x-1\rangle = ax^3 + (b-a)x^2 + (c-b)x - c$$

$$15a + 20(b-a) + 30(c-b) - 60c = 0$$

$$+5a + 10b + 30c = 0$$

$$60\langle ax^2+bx+c, x^2-x\rangle = ax^4 + (b-a)x^3 + (c-b)x^2 - cx$$

$$12a + 15(b-a) + 20(c-b) - 30c$$

$$3a + 5b + 10c = 0$$

$$\begin{bmatrix} 3 & 5 & 10 \\ 5 & 10 & 30 \end{bmatrix} \xrightarrow[②\leftarrow ② - 2①]{} \begin{bmatrix} 3 & 5 & 10 \\ -1 & 0 & 10 \end{bmatrix}\begin{bmatrix} 10 \\ \cancel{?} \\ 1 \end{bmatrix} = 0 \quad \begin{bmatrix} 10 \\ -8 \\ 1 \end{bmatrix} \text{ is in nullspace } \to 0$$

$\boxed{\text{so } 10x^2-8x+1 \text{ is } \perp \text{ to } W}$

(we also saw this in our check for our first try)

5)

$$x+1 = \underbrace{t(10x^2-8x+1)}_{\perp \text{ to } W} + \Big[\underbrace{(x+1)-t(10x^2-8x+1)}_{\text{want in } W}\Big]$$

we find t by plugging in $x=1$

$$2 = t(3) + (0)$$

this needs to be 0 to belong to W

$$\Rightarrow t = 2/3$$

$$(x+1) - 2/3(10x^2-8x+1)$$

$$= \underbrace{\boxed{-\frac{20}{3}x^2 + \frac{19}{3}x + 1/3}}_{\text{projection of } x+1 \text{ onto } W}$$