

F14 Homework 4 Problem 1

Linear Algebra, Dave Bayer

[1] Find A^n where A is the matrix

$$A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$A^n = \frac{\begin{pmatrix} -2 \\ 1 \end{pmatrix}^n}{1} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + \frac{\begin{pmatrix} -1 \\ 1 \end{pmatrix}^n}{1} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\lambda = -2, -1 \quad A^n = (-2)^n \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + (-1)^n \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$\begin{aligned} r+s &= -3 & -2, -1 \\ rs &= 2 \end{aligned}$$

$$\begin{array}{l} -2 \quad [2 \ 1] \quad \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{matrix} [2 \ 1] \\ [1 \ 1] \end{matrix} /_1 = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \quad \begin{matrix} -(A-\lambda_2) \checkmark \\ (A-\lambda_1) \checkmark \\ \lambda_2 - \lambda_1 = 1 \checkmark \end{matrix} \\ -1 \quad [1 \ 1] \quad \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix} \begin{matrix} [1 \ 1] \\ [1 \ 1] \end{matrix} /_1 = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark \end{array}$$

check answer in box for $n=1$

$$\begin{array}{c|c} -4 & -2 \\ \hline 1 & 1 \\ \hline 4 & -2 \\ \hline 2 & -2 \end{array} = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} = A \checkmark$$

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[2] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} -2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$e^{At} = \frac{e^{-t}}{2} \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + \frac{e^t}{2} \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}$$

$$\lambda = -1, 1 \quad e^{At} = \frac{e^{-t}}{2} \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + \frac{e^t}{2} \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix}$$

SUM = 0
 prod = $-4 + 3 = 1$ $-1, 1$

$$\begin{array}{l} -1 \quad [1 \ 1] \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \\ 1 \quad [1 \ 3] \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array} \quad \begin{array}{l} \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} / 2 \\ \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} / 2 \end{array} = \begin{array}{l} \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} / 2 \\ \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} / 2 \\ \hline \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} / 2 = I \end{array} \quad \begin{array}{l} -(A-I) / 2 \quad \checkmark \\ (A+I) / 2 \quad \checkmark \\ 2 = 1 - (-1) \quad \checkmark \end{array}$$

check answer in box

$$A = \frac{d}{dt} \frac{AAt}{t=0} = \frac{-1}{2} \begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 & -3 \\ 1 & 3 \end{bmatrix} = \begin{array}{c|c} -3 & -3 \\ \hline 1 & 3 \end{array} / 2 = \begin{bmatrix} -4 & -6 \\ 2 & 4 \end{bmatrix} / 2 \quad \checkmark$$

F14 Homework 4 Problem 3

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[3] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y = \frac{e^{-2t}}{1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\cancel{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}}{\cancel{1}} \begin{bmatrix} \cancel{0} \\ \cancel{0} \end{bmatrix}$$

$$\lambda = -2, 0 \quad e^{At} = \frac{e^{-2t}}{2} \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \quad y = \frac{e^{-2t}}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

sum = -2 -2, 0
 prod = 0

$-2 \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} / 2$ ↷ - ✓

$0 \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} / 2$ $0 - (-2) = 2 \checkmark$

I ✓

$$y = e^{At} y(0) = \frac{e^{-2t}}{2} \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

check $y' = -2e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-2t} \begin{bmatrix} -2 \\ -2 \end{bmatrix}$ ↷ = ✓

$Ay = e^{-2t} \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-2t} \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

F14 Homework 4 Problem 4

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[4] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}$$

$$e^{At} = \frac{e^{3t}}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{te^{3t}}{1} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

SUM = 6
prod = 9

3, 3

$$\lambda = 3, 3 \quad e^{At} = e^{3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{3t} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

$$A = 3I + N \quad N = A - 3I \quad N^2 = 0$$

$$e^{At} = e^{(3I+N)t} = e^{3It} e^{Nt} = e^{3t}(I + tN) = e^{3t}I + te^{3t}N$$

$$e^{At} = e^{3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{3t} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

check: $\frac{d}{dt} e^{At} = A e^{At} ?$

$$\underbrace{3e^{3t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (e^{3t} + 3te^{3t}) \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}}_{e^{3t} \left(3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \right) + te^{3t} \begin{bmatrix} 6 & -12 \\ 3 & -6 \end{bmatrix}} = \underbrace{e^{3t} \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\checkmark} + \underbrace{te^{3t} \begin{bmatrix} 5 & -4 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 6 & -12 \\ 3 & -6 \end{bmatrix}} \underbrace{\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}}_{\checkmark}$$

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[5] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{At} = \frac{e^{-t}}{4} \begin{bmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^t}{2} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} + \frac{e^{3t}}{4} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda = -1, 1, 3 \quad e^{At} = \frac{e^{-t}}{4} \begin{bmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{e^t}{2} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} + \frac{e^{3t}}{4} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} r+s+t &= 3 \quad \checkmark \\ rs+rt+st &= \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -3+1+1 = -1 \quad \checkmark \\ rst &= 1 \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = -3 \quad \checkmark \end{aligned}$$

A is block-triangular.
eigenvalues of blocks
[1] $\Rightarrow 1$, $\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$ $r+s=2$ $-1, 3$
 $rs=-3$

$-1, 1, 3$

$$\begin{array}{l} -1 \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ 1 \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \\ 3 \begin{bmatrix} -2 & 2 & 2 \\ 2 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \end{array} \quad \begin{array}{l} \begin{bmatrix} 2 & -2 & -1 \\ -2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} /4 \\ \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{bmatrix} /2 \\ \begin{bmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} /4 \end{array}$$

$$\begin{array}{c|c|c} 2 & -2 & -1 \\ 2 & 2 & -2 \\ \hline -2 & 2 & 1 \\ 2 & 2 & -4 \end{array} \begin{array}{c} 3 \\ -3 \\ 4 \end{array} = I \quad \checkmark$$

check $\frac{d}{dt} e^{At} \Big|_{t=0} = A?$

$$\begin{array}{c|c|c} -2 & 2 & -1 \\ 2 & -2 & 1 \\ \hline & & 4 \end{array} /4 = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} /4 = A \quad \checkmark$$

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[6] Find e^{At} where A is the matrix

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$e^{At} = \frac{e^{3t}}{9} \begin{bmatrix} 1 & 5 & 3 \\ 1 & 5 & 3 \\ 1 & 5 & 3 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 8 & -5 & -3 \\ -1 & 4 & -3 \\ -1 & -5 & 6 \end{bmatrix} + \frac{t}{3} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\lambda = 3, 0, 0 \quad e^{At} = \frac{e^{3t}}{9} \begin{bmatrix} 1 & 5 & 3 \\ 1 & 5 & 3 \\ 1 & 5 & 3 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 8 & -5 & -3 \\ -1 & 4 & -3 \\ -1 & -5 & 6 \end{bmatrix} + \frac{t}{3} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

A is singular $\Rightarrow r=0$ $s+t = \text{trace}(A) = 2+1=3$
 $s, t = 0, 3$ $st = \text{huh}(A) = \begin{vmatrix} 0 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 0 + 1 + 1 = 0$

$[0, 0, 3]$

$A - 3I: [1 \ 5 \ 3] \begin{bmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$A = 3B + 0C + N$
 $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

plan: find B
 $C = I - B$
 $N = A - 3B$

000
 (dream coordinates)

plan works in dream coords or original coords

$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \ 5 \ 3] / 9$
 $= \begin{bmatrix} 1 & 5 & 3 \\ 1 & 5 & 3 \\ 1 & 5 & 3 \end{bmatrix} / 9$

$C = \begin{bmatrix} 8 & -5 & -3 \\ -1 & 4 & -3 \\ -1 & -5 & 6 \end{bmatrix} / 9$ $N = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix} / 3$

checks: $BN = 0 \checkmark$ $N^2 = 0 \checkmark$

$$e^{At} = \frac{e^{3t}}{9} \begin{bmatrix} 1 & 5 & 3 \\ 1 & 5 & 3 \\ 1 & 5 & 3 \end{bmatrix} + \frac{e^{0t}}{9} \begin{bmatrix} 8 & -5 & -3 \\ -1 & 4 & -3 \\ -1 & -5 & 6 \end{bmatrix} + \frac{te^{0t}}{3} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 0 \end{bmatrix}$$

$\rightarrow + = I \checkmark$

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[7] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$y = \frac{e^{4t}}{9} \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} + \frac{e^t}{9} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} + \frac{te^t}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 4, 1, 1 \quad e^{At} = \frac{e^{4t}}{9} \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 6 & 4 & 2 \end{bmatrix} + \frac{e^t}{9} \begin{bmatrix} 6 & -2 & -1 \\ -6 & 5 & -2 \\ -6 & -4 & 7 \end{bmatrix} + \frac{te^t}{3} \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$y = \frac{e^{4t}}{9} \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} + \frac{e^t}{9} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} + \frac{te^t}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$r+s+t=6$$

$$rst = 2 \cdot \frac{2}{2} \cdot 1 - 1 \cdot \frac{2}{2} \cdot 1 = 2 \cdot 3 - 1 \cdot 2 = 4 \text{ factors } -4, -2, -1, 1, 2, 4$$

try 1:

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \text{ singular } r=1$$

$$s+t=5 \Rightarrow 1, 4 \quad (1, 1, 4)$$

$$\begin{bmatrix} 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} + 4 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

A C B N

plan: Find B. $C = I - B$. $N = A - 4B - C$

$$4 \begin{bmatrix} 3 & 2 & 1 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$B = \frac{1}{9} \begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & 2 \\ 6 & 4 & 2 \end{bmatrix} \quad C = \frac{1}{9} \begin{bmatrix} 6 & -2 & -1 \\ 6 & 5 & -2 \\ 6 & -4 & 7 \end{bmatrix}$$

$$N = A - 4B - C$$

$$N = \frac{1}{3} \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

checks $BN=0$ and $N^2=0$

$$\begin{array}{r|l|l} 18-12 & 9-8+2 & -4+1 \\ -6 & & \\ \hline 18-24 & 18-16+5 & 9-8+2 \\ +6 & & \\ \hline 18-24 & 9-16+4 & 18-8 \\ +6 & & -7 \end{array} \frac{1}{9}$$

$$= \frac{1}{9} \begin{bmatrix} 0 & 3 & -3 \\ 0 & -3 & 3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$y = \frac{e^{4t}}{9} B \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{e^t}{9} C \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{te^t}{3} N \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$y = \frac{e^{4t}}{9} \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} + \frac{e^t}{9} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} + \frac{te^t}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

hw4p7 check.

$$y = \frac{e^{4t}}{9} \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} + \frac{e^t}{9} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} + \frac{te^t}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$y' = e^{4t} \frac{4}{9} \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} + e^t \frac{1}{9} \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} + e^t \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + te^t \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$
$$= e^{4t} \begin{bmatrix} 16 \\ 32 \\ 32 \end{bmatrix} / 9 + e^t \begin{bmatrix} -4 \\ 4 \\ 13 \end{bmatrix} / 9 + te^t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} / 3$$

$$Ay = e^{4t} A \begin{bmatrix} 4 \\ 8 \\ 8 \end{bmatrix} / 9 + e^t A \begin{bmatrix} -4 \\ 1 \\ 10 \end{bmatrix} / 9 + te^t A \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} / 3$$
$$= e^{4t} \begin{bmatrix} 16 \\ 32 \\ 32 \end{bmatrix} / 9 + e^t \begin{bmatrix} -7 \\ 4 \\ 13 \end{bmatrix} / 9 + te^t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} / 3$$

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F14 Homework 4 Problem 8
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[8] Solve the differential equation $y' = Ay$ where

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 2 \\ -1 & -2 & 3 \end{bmatrix}, \quad y(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$y = \frac{e^{2t}}{1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \frac{te^{2t}}{1} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \frac{t^2e^{2t}}{1} \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$$

$$\lambda = 2, 2, 2$$

$$e^{At} = e^{2t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + te^{2t} \begin{bmatrix} -1 & -2 & 1 \\ -2 & 0 & 2 \\ -1 & -2 & 1 \end{bmatrix} + \frac{t^2e^{2t}}{2} \begin{bmatrix} 4 & 0 & -4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

$$y = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \frac{t^2e^{2t}}{2} \begin{bmatrix} -4 \\ 0 \\ -4 \end{bmatrix}$$

$$r+s+t=6, \quad rst = 1 \left| \begin{array}{c} 22 \\ -23 \end{array} \right| + 2 \left| \begin{array}{c} -22 \\ -13 \end{array} \right| + 1 \left| \begin{array}{c} -22 \\ -1-2 \end{array} \right| = 16 - 8 = 8.$$

$$1 \cdot 10 + 2 \cdot (-4) + 1 \cdot 6$$

try $r=1$: $\begin{bmatrix} 0 & -2 & 1 \\ -2 & 1 & 2 \\ -1 & -2 & 2 \end{bmatrix}$

$r=2$: $\begin{bmatrix} -1 & -2 & 1 \\ -2 & 0 & 2 \\ -1 & -2 & 1 \end{bmatrix}$ singular.
 $r=2 \checkmark$

$s+t=4$
 $st=4 \Rightarrow s, t=2, 2$
 $\boxed{2, 2, 2}$

$N = A - 2I$

check: $N^2 = \begin{bmatrix} 4 & 0 & -4 \\ 0 & 0 & 0 \\ 4 & 0 & -4 \end{bmatrix}$ $N^3 = 0 \checkmark$ $A = 2I + N$, $N^3 = 0$
 $e^{At} = e^{2t} (I + tN + \frac{t^2N^2}{2})$

$$y = e^{At} y(0) = e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + t^2e^{2t} \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$$

$$y' = e^{2t} (2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}) + te^{2t} (2 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}) + t^2e^{2t} (2 \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix})$$

check $\left\{ \begin{array}{l} y' \\ Ay \end{array} \right. = e^{2t} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + te^{2t} \begin{bmatrix} -6 \\ 4 \\ -6 \end{bmatrix} + t^2e^{2t} \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$

F14 Homework 4 Problem 9
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[9] Express the quadratic form

$$3x^2 + 3y^2 + 2xz + 2yz + 2z^2$$

as a sum of squares of orthogonal linear forms.

$$\frac{1}{6} (x+y-2z)^2 + \frac{3}{2} (x-y)^2 + \frac{4}{3} (x+y+z)^2$$

$$\lambda = 1, 3, 4 \quad A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\frac{1}{6} (x+y-2z)^2 + \frac{3}{2} (x-y)^2 + \frac{4}{3} (x+y+z)^2$$

$$[x \ y \ z] \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} r+s+t &= 8 \\ rst &= 3 \left| \begin{smallmatrix} 3 & 1 \\ 1 & 2 \end{smallmatrix} \right| - |0 \ 3| = 15 - 3 = 12 \\ \text{try } r=1 &\checkmark \quad s+t=7 \Rightarrow s, t=3, 4 \\ st &= 12 \end{aligned}$$

$$\begin{array}{l} 1 \\ 3 \\ 4 \end{array} \begin{array}{|c|} \hline \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \\ \hline \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \hline \end{array}$$

$$A = \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} + \frac{4}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$[x \ y \ z] A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6} (x+y-2z)^2 + \frac{3}{2} (x-y)^2 + \frac{4}{3} (x+y+z)^2$$

check x6

	x^2	y^2	z^2	xy	xz	yz
$(x+y-2z)^2$	1	1	4	2	-2	-2
$9(x-y)^2$	9	9	0	-18	0	0
$8(x+y+z)^2$	8	8	8	16	16	16

(check caught copying error -2 for -2z)

$$\div 6 \hookrightarrow \begin{array}{cccccc} 18 & 18 & 12 & 0 & 12 & 12 \\ 3 & 3 & 2 & 0 & 2 & 2 \end{array} \checkmark$$