F14 Homework 3 Problem 1
Linear Algebra, Dave Bayer

[1] Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\boxed{\begin{array}{cccc} + & & & \\ - & & & \\ + & & & \\ - & + & - & + \end{array}}$$

det(A) = $\boxed{-6}$

Expand by minors along last row:

$$+1\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \underbrace{\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}}_{2} - 2\underbrace{\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}}_{0} + 4\underbrace{\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix}}_{-2} = 2-8=-6$$

check: subtract col 1 from col 3 and col 4:

$$\begin{vmatrix} 1 & 2 & 1 & 3 \\ 1 & 3 & 2 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 3 & 1 & 1 & 2 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -3\cdot 2\cdot 1\cdot 1 = -6 \checkmark$$

two col swaps same sign

3 col swaps negates

triangular

F14 Homework 3 Problem 2
Linear Algebra, Dave Bayer

[2] Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 2 & 1 \\ 5 & 1 & 4 & 1 & 1 \\ 1 & 2 & 1 & 3 & 1 \\ 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

$\det(A) = \boxed{18}$

subtract last col from other cols

$$\begin{vmatrix} 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 \\ 4 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 4 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix} = -3\begin{vmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix}$$

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check, pull 3 out from last row then subtract from rest:

$$3\begin{vmatrix} 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 3\left(2\underbrace{\begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{vmatrix}}_{3\cdot 3} + 1\underbrace{\begin{vmatrix} 0 & 2 & 1 \\ 4 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix}}_{-4\cdot 3}\right) = 3(18-12) = 18$$

expand last col
then first row

$$= 3(6-4)(4-1) = 3\cdot 2\cdot 3$$
$$= 18$$

F14 Homework 3 Problem 3
Linear Algebra, Dave Bayer

[3] Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 1 \end{bmatrix} \Big\} \text{ check } AA^{-1} = I \checkmark$$

$$\boxed{A^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -2 & 0 \\ -1 & -1 & 5 \\ -2 & 3 & 0 \end{bmatrix}}$$

$$\begin{array}{c|cccc} & 3 & 2 & 1 & 3 \; 2 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 2 & 3 \\ 3 & 2 & 1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 0 \end{array}$$

$$\begin{bmatrix} +3 & -2 & 0 \\ -1 & -1 & +5 \\ -2 & +3 & 0 \end{bmatrix} \Big/ +5$$

note:

fix -5 by making every $-$ a $+$
then put in $-$ elsewhere
ignore zeros.

$$\begin{bmatrix} -3 & 2 & 0 \\ 1 & 1 & -5 \\ 2 & -3 & 0 \end{bmatrix} \Big/ -5 \implies \underbrace{\begin{bmatrix} +3 & 2 & 0 \\ 1 & 1 & +5 \\ 2 & +3 & 0 \end{bmatrix} \Big/ +5}_{} \implies \begin{bmatrix} +3 & -2 & 0 \\ -1 & -1 & +5 \\ -2 & +3 & 0 \end{bmatrix} \Big/ +5$$

if we get distracted here,
we can figure out where we ~~left~~ off,
and pick up from there.

F14 Homework 3 Problem 4
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[4] Using Cramer's rule, solve for x in the system of equations

$$\begin{bmatrix} 3 & a & 2 \\ 2 & b & 3 \\ 1 & c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{x = \frac{(5)\,a + (-3)\,b + (-1)\,c}{(1)\,a + (1)\,b + (-5)\,c}}$$

$$x = \frac{\begin{vmatrix} 1 & a & 2 \\ 1 & b & 3 \\ 2 & c & 1 \end{vmatrix}}{\begin{vmatrix} 3 & a & 2 \\ 2 & b & 3 \\ 1 & c & 1 \end{vmatrix}} = \frac{-a\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} + b\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{-a\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + b\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - c\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}}$$

$$= \frac{5a - 3b - c}{a + b - 5c}$$

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check:

$$\begin{bmatrix} 3 & -4 & 2 \\ 2 & -4 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(invent a sample case where we know the numbers)

pick easy random test values

what works here for a, b, c?

$$(a,b,c) = (-4,-4,0) \Rightarrow x = 1$$

$$x = \frac{5(-4) - 3(-4) - (0)}{(-4) + (-4) - 5(0)} = \frac{-8}{-8} = 1 \quad \boxed{\checkmark}$$

F14 Homework 3 Problem 5
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[5] Find the characteristic equation and a system of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}$$

$$\lambda^2 - \underset{(1-2)}{\mathrm{trace}(A)}\,\lambda + \underset{(1\cdot(-2)-1\cdot 4)}{\det(A)} = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$(\lambda+3)(\lambda-2) = 0$$

$$\lambda = 2, -3$$

$$\lambda^2 + (\boxed{1})\lambda + (\boxed{-6}) = 0$$

$$\lambda_1,\ \lambda_2 = \boxed{2},\ \boxed{-3}$$

$$v_1,\ v_2 = \begin{bmatrix} \boxed{4} \\ \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{1} \\ \boxed{-1} \end{bmatrix}$$

$\lambda = 2$ } $A - 2I$: $\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 0$

$\lambda = -3$ $A + 3I$: $\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

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check $\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ ✓

$\begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ✓

F14 Homework 3 Problem 6
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[6] Find the characteristic equation and a system of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 0 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\lambda^3 - \underset{2-2-1}{\mathrm{trace}(A)}\lambda^2 + \underbrace{\mathrm{huh}(A)}_{\begin{vmatrix} 2 & 1 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} -2 & 0 \\ -1 & -1 \end{vmatrix}}\lambda - \underset{-2\begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix}}{\det(A)} = 0$$

$$\quad -4 \qquad 0 \qquad 2$$

$$\lambda^3 + \lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda-1)(\lambda+2) = 0$$

$$\lambda = 0, 1, -2$$

$$\lambda^3 + (\boxed{1})\lambda^2 + (\boxed{-2})\lambda + (\boxed{0}) = 0$$

$$\lambda_1, \lambda_2, \lambda_3 = \boxed{0}, \boxed{1}, \boxed{-2}$$

$$v_1, v_2, v_3 = \begin{bmatrix} \boxed{1} \\ \boxed{0} \\ \boxed{2} \end{bmatrix}, \begin{bmatrix} \boxed{1} \\ \boxed{0} \\ \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{0} \\ \boxed{1} \\ \boxed{1} \end{bmatrix}$$

$\lambda = 0$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 0 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 0$$

$\lambda = 1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 0 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$\lambda = -2$

check:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 0 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\lambda = 0 \quad \lambda = 1 \quad \lambda = -2$$

F14 Homework 3 Problem 7
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[7] Express $f(n)$ using a matrix power, and find $f(8)$, where

$$f(0) = -1, \quad f(1) = 2$$
$$f(n) \;=\; f(n-1) \;+\; f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$f(8) = 29$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \det = -1$$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad \det = 1 = (-1)^2 \checkmark$$

$$A^4 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \quad \det = 1 \checkmark$$

$$A^8 = \begin{bmatrix} 13 & 21 \\ 21 & 34 \end{bmatrix} \quad \underbrace{13\cdot 34}_{\substack{340 \\ 102}} - \underbrace{21\cdot 21}_{441} = 1 \checkmark$$

$$\begin{bmatrix} f(8) \\ n \end{bmatrix} = \begin{bmatrix} 13 & 21 \\ n & n \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 42 - 13 = 29 \checkmark$$

0	-1	+
1	2	+
2	1	
3	3	
4	4	
5	7	
6	11	
7	18	
8	29 ✓	

F14 Homework 3 Problem 8
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[8] Express $f(n)$ using a matrix power, and find $f(8)$, where

$$f(0) = 1, \quad f(1) = 1, \quad g(1) = 1$$
$$f(n) = f(n-1) + g(n-1)$$
$$g(n) = f(n-1) + f(n-2)$$

A det=1

$$\begin{bmatrix} f(n) \\ f(n+1) \\ g(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A}^{n} \begin{bmatrix} f(0) \\ f(1) \\ g(1) \end{bmatrix}$$

$$f(8) = \boxed{81}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad \det = 1 \ \checkmark$$

$$A^4 = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 4 \\ 2 & 6 & 3 \end{bmatrix} \quad 1\underbrace{\begin{vmatrix} 7 & 4 \\ 6 & 3 \end{vmatrix}}_{-3} - 2\underbrace{\begin{vmatrix} 4 & 2 \\ 6 & 3 \end{vmatrix}}_{0} + 2\underbrace{\begin{vmatrix} 4 & 2 \\ 7 & 4 \end{vmatrix}}_{2} = 1 \ \checkmark$$

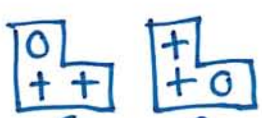
$$A^8 = \begin{bmatrix} 13 & 44 & 24 \\ \ast & \ast & \ast \\ \ast & \ast & \ast \end{bmatrix}$$

$1\cdot1 + 4\cdot2 + 2\cdot2 = 13$
$1\cdot4 + 4\cdot7 + 2\cdot6 = 4 + 28 + 12 = 44$
$1\cdot2 + 4\cdot4 + 2\cdot3 = 2 + 16 + 6 = 24$

$$f(8) = \begin{bmatrix} 13 & 44 & 24 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 81$$

check:

n	f(n)	g(n)
0	1	
1	1	1
2	2	2
3	4	3
4	7	6
5	13	11
6	24	20
7	44	37
8	81 ✓	



f g

sliding patterns for recursion

F14 Homework 3 Problem 9
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[9] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\begin{array}{cccccc} 1 & 1 & 0 & -1 & -1 & 0 \\ [\,] & [1] & \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Find $f(0)$ and $f(1)$. Find a recurrence relation for $f(n)$. Express $f(n)$ using a matrix power. Find $f(8)$.

$f(n) =$

$f(n-1) + f(n-2)$

$$f(0) = \boxed{1} \qquad f(1) = \boxed{1}$$

$$f(n) = \left(\boxed{1}\right) f(n-1) + \left(\boxed{-1}\right) f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{1} \\ \boxed{-1} & \boxed{1} \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$f(8) = \boxed{0}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A^8 \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} = \begin{bmatrix} f(8) \\ f(9) \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \leftarrow f(8)$$

n	f(n)
0	1
1	1
2	0
3	-1
4	-1
5	0
6	1
7	1
8	0
9	-1