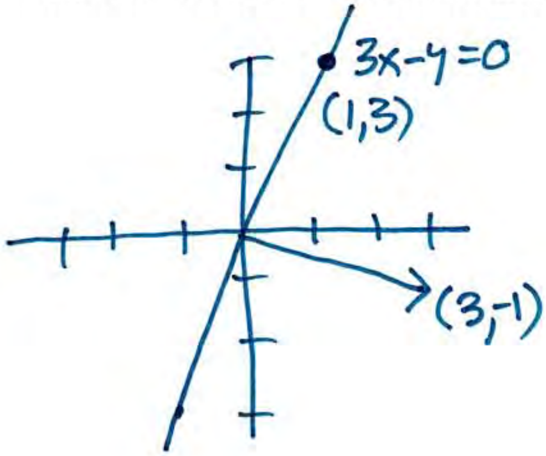
F14 Homework 2 Problem 1
Linear Algebra, Dave Bayer

[1] Find the 2×2 matrix which reflects across the line $3x - y = 0$.



$(1,3) \mapsto (1,3)$ fix

$(3,-1) \mapsto (-3,1)$ reflect

$$A \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} +1 & +3 \\ +3 & -1 \end{bmatrix} \Big/ {+10} = \begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix} \Big/ 10$$

$$\boxed{A = \begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \Big/ 5}$$

check: $\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix} \Big/ 5 \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -15 \\ 15 & 5 \end{bmatrix} \Big/ 5 = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ ✓

F14 Homework 2 Problem 2
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[2] Find the 3×3 matrix which vanishes on the plane $4x + 2y + z = 0$, and maps the vector $(1, 1, 1)$ to itself.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \end{bmatrix} \Big/ (1,1,1)\cdot(4,2,1) = \boxed{\begin{bmatrix} 4 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \Big/ 7}$$

check: $$\begin{bmatrix} 4 & 2 & 1 \\ 4 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \Big/ 7 \begin{bmatrix} 1 & \underbrace{1 & 0}_{} \\ 1 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 0 & 0 \\ 7 & 0 & 0 \\ 7 & 0 & 0 \end{bmatrix} \Big/ 7 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

vectors in plane ✓

F14 Homework 2 Problem 3
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[3] Find the 3×3 matrix which vanishes on the vector $(1, 1, 0)$, and maps each point on the plane $x + 2y + 2z = 0$ to itself.

$$A = I - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} / 3 = \begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix} / 3 - \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} / 3$$

$$= \boxed{\begin{bmatrix} 2 & -2 & -2 \\ -1 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix} / 3}$$

check: $$\begin{bmatrix} 2 & -2 & -2 \\ -1 & 1 & -2 \\ 0 & 0 & 3 \end{bmatrix} / 3 \begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & 0 \\ 0 & \underbrace{0 & -1}_{\text{in plane}} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} / 3 \quad \checkmark$$

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[4] Find the 3×3 matrix that projects orthogonally onto the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} t$$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} / 14 = \boxed{\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix} / 14}$$

check $(1,-2,3) \longmapsto (14,-28,42)/14 = (1,-2,3)$ ✓

$(2,1,0) \longmapsto 0$

$(0,3,2) \longmapsto 0$ ✓

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[5] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + 2y + 3z = 0$$

$$I - \begin{bmatrix}1\\2\\3\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}/14 = \begin{bmatrix}14&&\\&14&\\&&14\end{bmatrix}/14 - \begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}/14$$

$$= \boxed{\begin{bmatrix}13&-2&-3\\-2&10&-6\\-3&-6&5\end{bmatrix}/14}$$

check:

$$\begin{bmatrix}13&-2&-3\\-2&10&-6\\-3&-6&5\end{bmatrix}/14 \begin{bmatrix}1&2&3\\2&-1&0\\3&0&-1\end{bmatrix} = \begin{bmatrix}0&28&42\\0&-14&0\\0&0&-14\end{bmatrix}/14 \quad ✓$$

plane

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[6] Find the row space and the column space of the matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 & 10 \end{bmatrix}$$

② ← ② − ①
③ ← ③ − ①

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 5 & 9 \end{bmatrix}$$

③ ← ③ − 2②

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{rank } \underline{\underline{3}}$$

after row reduction, new rows span <u>same</u> rowspace but a <u>different</u> column space.

row space basis: $\left\{ \begin{matrix} [0\,0\,1\,1\,1\,1], \\ [0\,0\,0\,1\,2\,3], \\ [0\,0\,0\,0\,1\,3] \end{matrix} \right\}$

column space basis: $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}}_{\text{check: are these independent?}} \right\}$

transpose: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ <u>yes</u> ✓

exactly like middle columns in first row reduction.

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[7] By least squares, find the equation of the form $y = ax + b$ that best fits the data

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Big/ 6$$

$$\boxed{y = x + 1/3}$$

check:

x_i	y_i	$x + 1/3$	Δ
0	1	1/3	-2/3
1	0	4/3	4/3
2	3	7/3	-2/3

✓

$\perp$ to columns of A

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[8] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,2,0,0) \qquad (0,1,2,0) \qquad (1,3,3,2) \qquad (0,0,1,2) \qquad (1,3,3,2)$$

Extend this basis to an orthogonal basis for $\mathbb{R}^4$.

V is space $8w+4x+2y+z=0$

$(1,2,0,0)$
$(0,0,1,2)$ from list

$(2a,-a,2b,-b)$ will be $\perp$ to above, for any a,b.

$(8,4,2,1)\cdot(2a,-a,2b,-b) = 12a+3b=0$

$a=1,\ b=-4$

$(2,-1,-8,4)$

$$\boxed{\mathbb{R}^4 \left\{ \begin{array}{l} V \left\{ \begin{array}{l} (1,2,0,0) \\ (0,0,1,2) \\ (2,-1,-8,4) \end{array} \right. \\ \quad (8,-4,2,-1) \end{array} \right.}$$

check

$$\begin{array}{rrrr} 2 & -1 & -8 & 4 \\ \times\ 8 & 4 & 2 & 1 \\ \hline 16 & -4 & -16 & 4 \end{array} = 0$$

$(8,-4,2,-1)$ is $\perp$ to each vector in V

F14 Homework 2 Problem 9
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[9] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace consisting of those polynomials $f(x)$ such that $f(-1) = 0$. Find the orthogonal projection of the polynomial $x + 1$ onto the subspace W, with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$$

$f(x) = x+1$

$f(-1) = 0 \Rightarrow x+1$ belongs to W

$\Rightarrow$ projection to W is self:

$$\boxed{x+1}$$