F14 Homework 1 Problem 1
Linear Algebra, Dave Bayer

[1] Solve the following system of equations.

$$\begin{bmatrix} 2 & -3 & -1 & 1 \\ 1 & -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

Check:

$$\begin{bmatrix} 2 & -3 & -1 & 1 \\ 1 & -2 & -2 & 0 \end{bmatrix} \left(\begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & -3 & -1 & 1 & 3 \\ 1 & -2 & -2 & 0 & 2 \end{array}\right] \xrightarrow[\text{①} \leftarrow \text{①} - 2\text{②}]{} \left[\begin{array}{cccc|c} 0 & 1 & 3 & 1 & -1 \\ 1 & -2 & -2 & 0 & 2 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 3 & 1 \\ 1 & -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 1 \\ -2 \end{bmatrix} x + \begin{bmatrix} 3 \\ -2 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} w + \begin{bmatrix} 1 \\ 0 \end{bmatrix} z = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} x + \begin{bmatrix} -3 \\ 2 \end{bmatrix} y$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} z = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix} y$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \\ -3 \end{bmatrix} t}$$

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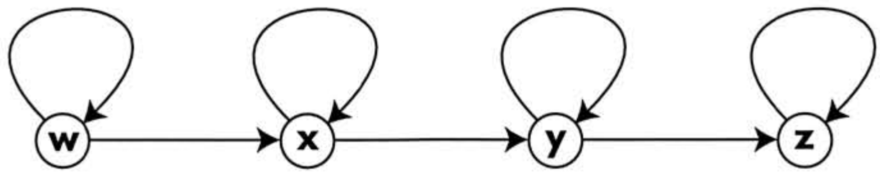
[2] Using matrix multiplication, count the number of paths of length ten from w to y.



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45 paths

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad A^{10} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 10 & 1 & 0 & 0 \\ 45 & 10 & 1 & 0 \\ 120 & 45 & 10 & 1 \end{bmatrix}$$

Can ignore z, no way to leave once entered

$$\begin{array}{cc} & \overbrace{\begin{array}{ccc} w & x & y \end{array}}^{\text{in}} \\ \text{out}\left[\begin{array}{c} w \\ x \\ y \end{array}\right. & \boxed{\begin{array}{ccc} 1 & & \\ 1 & 1 & \\ 0 & 1 & 1 \end{array}} \end{array} = A$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 2 & 1 \end{bmatrix}$$

$$A^4 = A^2 \cdot A^2 = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 6 & 4 & 1 \end{bmatrix}$$

$$A^8 = A^4 \cdot A^4 = \begin{bmatrix} 1 & & \\ 4 & 1 & \\ 6 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 4 & 1 & \\ 6 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 8 & 1 & \\ 28 & 8 & 1 \end{bmatrix}$$

$$A^{10} = A^2 \cdot A^8 = \begin{bmatrix} & & \\ & & \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ 8 & & \\ 28 & & \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ 45 & & \end{bmatrix}$$

don't need other entries

45 paths

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[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix}0&1&12\\1&0&0\\0&0&3\end{bmatrix}$$

$$\begin{bmatrix}0&1&12\\1&0&0\\0&0&3\end{bmatrix} = \begin{bmatrix}1&0&4\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

$$\begin{tikzcd}[column sep=huge]
\begin{bmatrix}0&1&12\\1&0&0\\0&0&3\end{bmatrix} \arrow[r, bend left, "(1)\leftrightarrows(2)"] & \begin{bmatrix}1&0&0\\0&1&12\\0&0&3\end{bmatrix} \arrow[l, bend left, "(1)\leftrightarrows(2)"] \arrow[r, bend left, "(3)\leftarrow\frac{1}{3}(3)"] & \begin{bmatrix}1&0&0\\0&1&12\\0&0&1\end{bmatrix} \arrow[l, bend left, "(3)\leftarrow 3(3)"] \arrow[r, bend left, "(2)\leftarrow(2)-12(3)"] & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \arrow[l, bend left, "(2)\leftarrow(2)+12(3)"]
\end{tikzcd}$$

$$\boxed{A = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&12\\0&0&1\end{bmatrix}}$$

$$\underbrace{\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&12\\0&0&1\end{bmatrix}}_{\begin{bmatrix}1&0&0\\0&1&12\\0&0&3\end{bmatrix}}}_{\begin{bmatrix}0&1&12\\1&0&0\\0&0&3\end{bmatrix}}$$

$\checkmark$

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[4] Find the matrix A such that

$$A \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

$AB = C \Leftrightarrow ABB^{-1} = CB^{-1} \Rightarrow A = CB^{-1}$

find B^{-1} row reducing $[B|I]$ to $[I|B^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad ② \leftarrow ② + ①$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \quad ③ \leftarrow ③ + ②$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right] \quad ③ \leftarrow \tfrac{1}{3}③ \text{ (combine over single denom)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & 0 & 0 \\ 0 & 1 & 2 & 3 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]_{/3} \quad \begin{array}{l} ① \leftarrow ① - ③ \\ ② \leftarrow ② - 2③ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]_{/3}$$

check: $\begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}_{/3} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{/3} = I$ ✓

$CB^{-1} =$

$$\begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix}_{/3} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}_{/3}$$

$$\boxed{A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$$

F14 Homework 1 Problem 5
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[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

plug in
solve for r,s

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} t$$

Check:

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} t \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} t$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} t \right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} t = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix} t$$