F14 8:40 Exam 3 Problem 1
Linear Algebra, Dave Bayer

[1] Find the determinant of the matrix

$$\begin{bmatrix} 4 & 6 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 6 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\det(A) = \boxed{-10}$

Subtract row 4 from other rows:

$$\begin{vmatrix} 3 & 5 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} \quad +1\left(-5\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\right) = -10$$

Check:
subtract col 4 from other columns:

$$\begin{vmatrix} 3 & 5 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = +1\begin{vmatrix} 3 & 5 & 1 \\ 1 & 0 & 1 \\ 0 & 5 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 \\ 5 & 0 \end{vmatrix} - 1\begin{vmatrix} 5 & 1 \\ 5 & 0 \end{vmatrix}$$

$$= -15 + 5 = -10 \ ☑$$

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[2] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

check $AA^{-1} = I$

$$A^{-1} = \frac{1}{\boxed{2}} \begin{bmatrix} \boxed{-1} & \boxed{0} & \boxed{1} \\ \boxed{-1} & \boxed{-4} & \boxed{3} \\ \boxed{2} & \boxed{2} & \boxed{-2} \end{bmatrix}$$

$$\begin{array}{c|cccc} 1 & 2 & 3 & 1 & 2 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 & 0 \end{array}$$

$$A^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & -4 & 3 \\ 2 & 2 & -2 \end{bmatrix} \Big/ 2$$

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[3] Using Cramer's rule, solve for z in the system of equations

$$\begin{bmatrix} 1 & a & 1 \\ 2 & b & 3 \\ 1 & c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\boxed{z = \frac{(0)\,a + (-1)\,b + (2)\,c}{(1)\,a + (0)\,b + (-1)\,c}}$$

$$z = \frac{\begin{vmatrix} 1 & a & 2 \\ 2 & b & 2 \\ 1 & c & 1 \end{vmatrix}}{\begin{vmatrix} 1 & a & 1 \\ 2 & b & 3 \\ 1 & c & 1 \end{vmatrix}} = \frac{-a\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + b\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}}{-a\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} + b\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - c\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{0a - 1b + 2c}{a + 0b - c}$$

check $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} 1 + \begin{bmatrix} -1 \\ -6 \\ -2 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} 2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

pick sample values for x, y, z, solve for a, b, c
to come up with sample solution

$$\left.\begin{matrix} a = -1 \\ b = -6 \\ c = -2 \end{matrix}\right\} \Rightarrow z = 2 \qquad z = \frac{-(-6) + 2(-2)}{(-1) - (-2)} = \frac{2}{1} = 2 \ ☑$$

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[4] Find the characteristic equation and a system of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix}$$

$$\lambda^2 + (4)\lambda + (3) = 0$$
$$\lambda_1,\ \lambda_2 = -1,\ -3$$
$$v_1,\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\lambda^2 - \underbrace{\text{trace}(A)}_{0+(-4)}\lambda + \underbrace{\det(A)}_{3} = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$(\lambda+1)(\lambda+3) = 0$$

$$\lambda = -1, -3$$

$$\lambda = -1: \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$$

$$\lambda = -3: \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 0$$

check:
$$\begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ 1 \end{bmatrix} ✓$$

$$\begin{bmatrix} 0 & -1 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 3 \end{bmatrix} ✓$$

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[5] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\underset{}{\overset{1}{[\,]}} \quad \overset{2}{\begin{bmatrix} 2 \end{bmatrix}} \quad \overset{3}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} \quad \overset{2\cdot 3 - 1\cdot 2 = 4}{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}} \quad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Find $f(0)$ and $f(1)$. Find a recurrence relation for $f(n)$. Express $f(n)$ using a matrix power. Find $f(8)$.

$$f(0) = \boxed{1} \qquad f(1) = \boxed{2}$$

$$f(n) = \left(\boxed{2}\right) f(n-1) + \left(\boxed{-1}\right) f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{1} \\ \boxed{-1} & \boxed{2} \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$f(8) = \boxed{9}$$

$$f(n) = \boxed{2 \quad \boxed{f(n-1)}} + \boxed{1 \;\; 1 \quad \boxed{f(n-2)}}$$

(every det term starts $\boxed{\bullet}$ or $\boxed{\cdot\bullet}$)

(or use expansion by minors)

$$f(n) = 2f(n-1) - f(n-2)$$

$$\begin{bmatrix} f(1) \\ f(2) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \quad \det = 1$$

$$A^2 = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad A^4 = \begin{bmatrix} -3 & 4 \\ -4 & 5 \end{bmatrix} \quad A^8 = \begin{bmatrix} -7 & 8 \\ -8 & 9 \end{bmatrix} \quad \begin{bmatrix} -7 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 9$$

det=1 ✓ det=1 ✓ det=1 ✓

check:

n	0	1	2	3	4	5	6	7	8
f(n)	1	2	3	4	5	6	7	8	9

$\boxed{-1 \;\; 2} \rightarrow$

sliding pattern