F14 11:40 Exam 3 Problem 1
Linear Algebra, Dave Bayer

[1] Find the determinant of the matrix

$$A = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 4 & 1 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 2 & 4 & 2 \end{bmatrix}$$

det(A) = -54

subtract first column from others

$$\begin{vmatrix} 3 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 0 & 3 \\ 2 & 0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 1 & 3 \end{vmatrix} \cdot \begin{vmatrix} 0 & 3 \\ 2 & 0 \end{vmatrix} = 9(-6) = -54$$

block triangular

check:
pull 3 out of first row, subtract multiples from other rows

$$3 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 0 \end{vmatrix} = 3 \cdot 1 \begin{vmatrix} 3 & 0 & 0 \\ 1 & 0 & 3 \\ 0 & 2 & 0 \end{vmatrix} = 3(-18) = -54 \; \checkmark$$

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[2] Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

check $AA^{-1} = I$ ☑

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -1 & 6 & -2 \\ -1 & 0 & 1 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\begin{array}{c|cccc|}
 & 1 & 1 & 1 & 1 \\
\hline
0 & 1 & 3 & 0 & 1 \\
2 & 1 & 2 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 3 & 0 & 1 \\
\hline
\end{array}$$

$$A^{-1} = \begin{bmatrix} -1 & 6 & -2 \\ -1 & 0 & 1 \\ 2 & -3 & 1 \end{bmatrix} \Big/ 3$$ ☑☑

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[3] Using Cramer's rule, solve for y in the system of equations

$$\begin{bmatrix} a & 1 & 2 \\ b & 1 & 3 \\ c & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\boxed{y = \frac{(-5)\,a + (3)\,b + (1)\,c}{(-2)\,a + (1)\,b + (1)\,c}}$$

$$y = \frac{\begin{vmatrix} a & 1 & 2 \\ b & 1 & 3 \\ c & 2 & 1 \end{vmatrix}}{\begin{vmatrix} a & 1 & 2 \\ b & 1 & 3 \\ c & 1 & 1 \end{vmatrix}} = \frac{a\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - b\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} + c\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}}{a\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} - b\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + c\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}} = \frac{-5a+3b+c}{-2a+b+c}$$

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check:

$$\begin{bmatrix} -3 \\ -4 \\ -1 \end{bmatrix} 1 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 2 + \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} 1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

pick simple values for x,y,z, solve for a,b,c
to come up with sample solution

$$\left.\begin{aligned} a &= -3 \\ b &= -4 \\ c &= -1 \end{aligned}\right\} \Rightarrow y = 2 \qquad y = \frac{-5(-3)+3(-4)+(-1)}{-2(-3)+(-4)+(-1)} = \frac{2}{1} = 2 \;\checkmark$$

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[4] Find the characteristic equation and a system of eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\lambda^2 + (\boxed{-3})\lambda + (\boxed{-4}) = 0$$

$$\lambda_1, \lambda_2 = \boxed{-1}, \boxed{4}$$

$$v_1, v_2 = \begin{bmatrix} \boxed{2} \\ \boxed{3} \end{bmatrix}, \begin{bmatrix} \boxed{1} \\ \boxed{-1} \end{bmatrix}$$

$$\lambda^2 - \underbrace{\mathrm{trace}(A)}_{2+1}\lambda + \underbrace{\det(A)}_{2-6} = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda + 1)(\lambda - 4) = 0$$

$$\lambda = -1, 4$$

$$\lambda = -1 : \begin{bmatrix} 3 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 0$$

$$\lambda = 4 : \begin{bmatrix} -2 & -2 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

check:

$$\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} = -1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \checkmark$$

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[5] Let $f(n)$ be the determinant of the $n \times n$ matrix in the sequence

$$\overset{1}{[\,]} \quad \overset{-2}{\begin{bmatrix} -2 \end{bmatrix}} \quad \overset{2}{\begin{bmatrix} -2 & 1 \\ 2 & -2 \end{bmatrix}} \quad \begin{bmatrix} -2 & 1 & 0 \\ 2 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \quad \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Find $f(0)$ and $f(1)$. Find a recurrence relation for $f(n)$. Express $f(n)$ using a matrix power. Find $f(8)$.

$$f(0) = \boxed{1} \qquad f(1) = \boxed{-2}$$

$$f(n) = \left(\boxed{-2}\right) f(n-1) + \left(\boxed{-2}\right) f(n-2)$$

$$\begin{bmatrix} f(n) \\ f(n+1) \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{1} \\ \boxed{-2} & \boxed{-2} \end{bmatrix}^n \begin{bmatrix} f(0) \\ f(1) \end{bmatrix}$$

$$f(8) = \boxed{16}$$

$f(n) = \boxed{(-2)\ \boxed{f(n-1)}} + \boxed{(2)^{(1)}\ \boxed{f(n-2)}}$

(every det term starts ⌐• or ⌐••)

(or use expansion by minors)

$f(n) = -2f(n-1) - 2f(n-2)$

$$\begin{bmatrix} f(1) \\ f(2) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}}^{A} \begin{bmatrix} f(0) \\ f(1) \end{bmatrix} \qquad \det = 2$$

$$A^2 = \begin{bmatrix} -2 & -2 \\ 4 & 2 \end{bmatrix} \quad A^4 = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} \quad A^8 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix} \qquad \begin{bmatrix} 16 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 16$$

$\det = 4$ ✓ $\qquad \det = 16$ ✓ $\qquad \det = 16^2$ ✓

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check:

n	0	1	2	3	4	5	6	7	8
$f(n)$	1	-2	2	0	-4	8	-8	0	16

✓

$\boxed{-2\ -2}$ sliding pattern