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Linear Algebra, Dave Bayer

[1] Find the 3×3 matrix which maps the vector $(0, 1, 1)$ to $(0, 2, 2)$, and maps each point on the plane $x + y = 0$ to the zero vector.

some multiple of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ will work.

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ so}$$

$$2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}}$$

check: $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ ✓

in plane

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[2] Find a basis for the row space and a basis for the column space of the matrix

these rows for row space

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 0 & -1 & -1 \end{bmatrix}$$

① = −2② − 2③ − 2④ is dependent so rank ≠ 4

rank 3 submatrix so rank ≥ 3

$\boxed{\text{rank} = 3}$

these columns for column space

basis for row space: $(-2,0,0,1,1,0)$, $(0,-2,0,-1,0,1)$, $(0,0,-2,0,-1,1)$

basis for column space: $(1,-2,0,0)$, $(1,0,-2,0)$, $(1,0,0,-2)$

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[3] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + 2y = 0$$

$$\underbrace{(1,2,0)}_{\text{normal vector to plane}} \cdot (x,y,z) = 0$$

$I = A + B$ A projects $\perp$ to plane
B projects $\perp$ onto line $(1,2,0)$

$$B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \Big/ (1,2,0)\cdot(1,2,0) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Big/ 5$$

$$A = I - B = \boxed{\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Big/ 5}$$

check
$$\begin{bmatrix} 4 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Big/ 5 \; \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Big/ 5 \quad ✓$$

$\perp$ in plane

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[4] Find an $\boxed{\text{orthogonal}}$ basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(1,-2,0,0) \qquad (1,0,-2,0) \qquad (1,0,0,-2) \qquad (0,1,-1,0) \qquad (0,1,0,-1) \qquad (0,0,1,-1)$$

Extend this basis to an $\boxed{\text{orthogonal}}$ basis for $\mathbb{R}^4$.

$$\left.\begin{matrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{matrix}\right\}$$ independent, start in different positions. So rank $V \geq 3$. $\boxed{\text{rank } V = 3}$

Every vector in V satisfies $2w+x+y+z=0$ so rank $V \leq 3$.

$(1,-2,0,0)$, $(0,0,1,-1)$ already $\perp$.

All of V is $\perp$ to $(2,1,1,1)$: $(2,1,1,1)\cdot(w,x,y,z)=0$

Find last vector for V: $(2a,a,b,b)$ is $\perp$ to $(1,-2,0,0)$ and $(0,0,1,-1)$

$(2a,a,b,b)\cdot(2,1,1,1) = 5a+2b=0 \qquad a=2,\ b=-5$

$(4,2,-5,-5)\cdot(2,1,1,1) = 8+2-5-5=0$ ✓

$$\boxed{\begin{array}{l} \left.\begin{array}{l}(1,-2,0,0)\\(0,0,1,-1)\\(4,2,-5,-5)\end{array}\right\}\ \perp \text{ basis for } V \\ \hline \left.(2,1,1,1)\right\}\ \text{extend to } \perp \text{ basis for } \mathbb{R}^4 \end{array}}$$

check: All pairs $\perp$ ✓

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[5] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of V consisting of those polynomials $f(x)$ such that the second derivative $f''(x) = 0$.

Find the orthogonal projection of the polynomial x^2 onto the subspace W, with respect to the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$$

$V = \{ax^2+bx+c\} \cong \{(a,b,c)\} = \mathbb{R}^3$

$W: \quad f(x) = ax^2+bx+c$

$\quad f''(x) = 2a = 0 \Rightarrow W = \{bx+c\} \cong \mathbb{R}^2$

Find $\perp$ basis for W. Want 1, and $bx+c \perp 1$

$\langle bx+c, 1\rangle = \int_0^1 (bx+c)dx = \frac{1}{2}b + c = 0 \Rightarrow \{1, 2x-1\}$

check: $\langle 2x-1, 1\rangle = \int_0^1 (2x-1)dx = 1-1 = 0$ ☑

Project x^2 onto this basis:

$$\frac{\langle x^2, 1\rangle}{\langle 1,1\rangle} = \frac{1/3}{1} = 1/3 \qquad \frac{\langle x^2, 2x-1\rangle}{\langle 2x-1, 2x-1\rangle} = \frac{\int_0^1 (2x^3-x^2)dx}{\int_0^1 (4x^2-4x+1)dx} = \frac{2/4 - 1/3}{4/3 - 4/2 + 1} = \frac{1/6}{1/3} = \frac{1}{2}$$

$$x^2 \mapsto 1/3(1) + 1/2(2x-1) = \boxed{x - 1/6}$$

check: Is $x^2 - (x - 1/6) \perp$ to both 1 and x?

$\langle x^2 - x + 1/6, 1\rangle = 1/3 - 1/2 + 1/6 = 0$ ☑

$\langle x^2 - x + 1/6, x\rangle = 1/4 - 1/3 + 1/12 = 0$ ☑