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Linear Algebra, Dave Bayer

[1] Find the 3×3 matrix which maps the vector $(1,1,1)$ to $(2,2,2)$, and maps each point on the plane $x+y+z=0$ to itself.

$\underbrace{}_{A}$

$A-I$ sends $(1,1,1)$ to $(1,1,1)$
vanishes on $x+y+z=0$

so $A-I = \begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1&1&1\end{bmatrix}/3 = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}/3$

$$A = \boxed{\begin{bmatrix}4&1&1\\1&4&1\\1&1&4\end{bmatrix}/3}$$

check: $\begin{bmatrix}4&1&1\\1&4&1\\1&1&4\end{bmatrix}/3 \begin{bmatrix}1&1&0\\1&-1&1\\1&0&-1\end{bmatrix} = \begin{bmatrix}6&3&0\\6&-3&3\\6&0&-3\end{bmatrix}/3$ ✓

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[2] Find a basis for the row space and a basis for the column space of the matrix

$$\begin{array}{l} \\ \text{these rows} \to \\ \text{for row space} \to \\ \\ \end{array}
\begin{bmatrix} -1 & -1 & 0 & -2 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 & -2 \\ -1 & -1 & 0 & -2 & 1 \end{bmatrix}
\begin{array}{l} ① = -② - ③ \\ ② \\ ③ \\ ④ = ① \end{array}$$

so rank ≤ 2

rank 2 submatrix so rank ≥ 2

these columns for columnspace

$\boxed{\text{rank} = 2}$

basis for row space: (1,0,1,1,1)
(0,1,-1,1,-2)

basis for column space: (-1,1,0,-1), (-1,0,1,-1)

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[3] Find the 3×3 matrix that projects orthogonally onto the plane

$$x + y - 2z = 0$$

$$\underbrace{(1,1,-2)}_{\text{normal vector to plane}} \cdot (x,y,z) = 0$$

$I = A + B$ A projects $\perp$ onto plane
B projects $\perp$ onto line $(1,1,-2)$

$$B = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \end{bmatrix} \Big/ (1,1,-2)\cdot(1,1,-2) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix} \Big/ 6$$

$$A = I - B = \boxed{\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 6}$$

check $$\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 6 \; \begin{bmatrix} \underbrace{\begin{matrix} 1 \\ 1 \\ -2 \end{matrix}}_{\perp} & \underbrace{\begin{matrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{matrix}}_{\text{in plane}} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & -6 & 12 \\ 0 & 0 & 6 \end{bmatrix} \Big/ 6$$ ✓

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[4] Find an $\boxed{\text{orthogonal}}$ basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(-1, 1, 0, -1) \qquad (-1, 0, 1, -1) \qquad (0, 1, -1, 0) \qquad (-2, 1, 1, -2) \qquad (1, 1, -2, 1)$$

Extend this basis to an $\boxed{\text{orthogonal}}$ basis for $\mathbb{R}^4$.

$$\begin{array}{cl} \begin{bmatrix} -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ -2 & 1 & 1 & -2 \\ 1 & 1 & -2 & 1 \end{bmatrix} & \begin{array}{l} \left.\begin{array}{l} 1 \\ 2 \end{array}\right\} \text{basis for } V \\ 1 - 2 \\ 1 + 2 \\ 1 - 2(2) \end{array} \end{array}$$

$\boxed{\text{rank } V = 2}$ Two independent conditions on V.

indeed, $(\overbrace{1, 1, -2}^{\text{first three sum to } 0}, 1)$, $(1, \underbrace{1, -2, 1}_{\text{last three sum to } 0})$

$$(-1,0,1,-1) - \frac{(-1,0,1,-1)\cdot(-1,1,0,-1)}{(-1,1,0,-1)\cdot(-1,1,0,-1)}(-1,1,0,-1)$$

$$= (-1,0,1,-1) - \tfrac{2}{3}(-1,1,0,-1) = (-\tfrac{1}{3}, -\tfrac{2}{3}, 1, -\tfrac{1}{3}) \sim (-1,-2,3,-1)$$

So $(-1,1,0,-1), (-1,-2,3,-1)$ is $\perp$ basis for V

$(1,1,1,0)$ and $(0,1,1,1)$ are both $\perp$ to V

$$(0,1,1,1) - \frac{(0,1,1,1)\cdot(1,1,1,0)}{(1,1,1,0)\cdot(1,1,1,0)}(1,1,1,0) = (0,1,1,1) - \tfrac{2}{3}(1,1,1,0)$$

$$\sim (-2,1,1,3)$$

$$\boxed{\begin{array}{l} \left.\begin{array}{l} (-1,1,0,-1) \\ (-1,-2,3,-1) \end{array}\right\} \perp \text{ basis for } V \\ \hline \left.\begin{array}{l} (1,1,1,0) \\ (-2,1,1,3) \end{array}\right\} \text{extend to } \perp \text{ basis for } \mathbb{R}^4 \end{array}}$$

check: All pairs $\perp$ ✓

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[5] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of V consisting of those polynomials $f(x)$ such that the $\boxed{\text{derivative}}$ $f'(0) = 0$.

Find the orthogonal projection of the polynomial x onto the subspace W, with respect to the inner product

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$$

$V = \{ax^2+bx+c\} \cong \{(a,b,c)\} = \mathbb{R}^3$

W: $f(x) = ax^2+bx+c$

$f'(x) = 2ax+b \quad f'(0) = b = 0 \Rightarrow W = \{ax^2+c\} \cong \mathbb{R}^2$

Gram-Schmidt $\{1, x^2\}$ into $\perp$ basis for W:

$x^2 \rightarrow x^2 - \dfrac{\langle x^2,1\rangle}{\langle 1,1\rangle} 1 = x^2 - 1/3 \sim 3x^2-1$

$\langle x^2, 1\rangle = \int_0^1 x^2 dx = 1/3$

$\langle 1,1\rangle = 1$

Check: $\langle 1, 3x^2-1\rangle = \int_0^1 (3x^2-1)dx = 1-1 = 0$ ☑

Now project x onto this basis:

$\dfrac{\langle x, 3x^2-1\rangle}{\langle 3x^2-1, 3x^2-1\rangle} = \dfrac{\int_0^1 (3x^3-x)dx}{\int_0^1 (9x^4-6x^2+1)dx} = \dfrac{(3/4 - 1/2)\,60}{(9/5-6/3+1)\,60} = \dfrac{15}{108-120+60} = \dfrac{15}{48} = \dfrac{5}{16}$

$\dfrac{\langle x,1\rangle}{\langle 1,1\rangle} = \dfrac{1/2}{1} = 1/2 \qquad \dfrac{1}{2}1 + \dfrac{5}{16}(3x^2-1) = \boxed{\dfrac{15}{16}x^2 + \dfrac{3}{16}}$

Check: Is $x - (\frac{15}{16}x^2 + 3/16)$ $\perp$ to W?

$\langle -\frac{15}{16}x^2 + x - 3/16, 1\rangle = -15/48 + 1/2 - 3/16 = \dfrac{-15+24-9}{48} = 0$ ☑

$\langle -\frac{15}{16}x^2 + x - 3/16, x^2\rangle = \dfrac{-15}{80} + \dfrac{1}{4} - 3/48 = \dfrac{-9+12-3}{48} = 0$ ☑

$\dfrac{-15}{80} = \dfrac{-3}{16} = \dfrac{-9}{48} \qquad \dfrac{1}{4} = \dfrac{12}{48}$