

[1] Solve the following system of equations.

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \\ 2 \end{bmatrix} t$$

Check:

$$\begin{bmatrix} 1 & 4 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \\ 2 \end{bmatrix} t \right) = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} t = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 4 & 0 & 1 & 4 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 3 \end{array} \right] \xrightarrow{\textcircled{1} \leftarrow \textcircled{1} - \textcircled{3}} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & \frac{1}{2} \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 3 \end{array} \right]$$

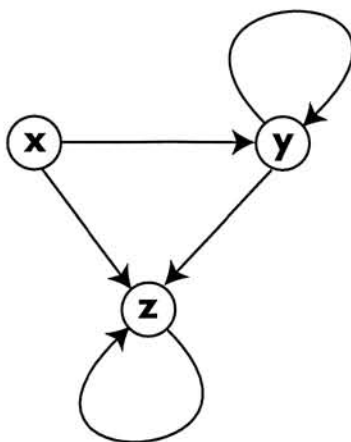
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} w + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} z = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 3 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ -2 \\ -2 \end{bmatrix} t}$$

particular solution $\mapsto \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$

homogeneous solution $\mapsto \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

[2] Using matrix multiplication, count the number of paths of length eight from x to z .



8 paths $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}, A^8 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 8 & 8 & 1 \end{bmatrix}$

in

at

$$\begin{matrix} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} x \\ y \\ z \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{matrix} = A$$

(5 entries = 5 edges)

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}$$

$$A^8 = \begin{matrix} x \\ z \end{matrix} \begin{bmatrix} 8 \end{bmatrix}$$

$\rightarrow (4, 4, 1) \cdot (0, 1, 4) = 0 + 4 + 4$

8 paths

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{2} \leftarrow \frac{1}{2} \textcircled{2}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{2} \leftarrow \textcircled{2} - \textcircled{1}} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\textcircled{1} \leftrightarrow \textcircled{2} I} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ +2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

do not include identity matrix
 many different solutions are correct

[4] Find the matrix A such that

$$A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Check:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \xrightarrow{\textcircled{2} \leftarrow \textcircled{2} - \textcircled{1}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \xrightarrow{\textcircled{3} \leftarrow \textcircled{3} - \textcircled{2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(column reduction)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

check: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \checkmark$

always check your work
 it will eventually become how you understand everything.

[5] Find the intersection of the following two affine subspaces of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

check
☒ soln satisfies these eqs

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}$$

☒ soln contained in this parametrized space

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix} t$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} t$$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} t \right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix} t$$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \\ -1 \end{bmatrix} t}$$