Exam 2

Linear Algebra, Dave Bayer, October 22, 2013

Name: ______________________________ **Uni:** ______________

[1]	[2]	[3]	[4]	[5]	**Total**

If you need more that one page for a problem, clearly indicate on each page where to look next for your work.

[1] Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$(2,0,1,0),\ (2,0,0,1),\ (0,2,1,0),\ (0,2,0,1)$$

Extend this basis to a basis for $\mathbb{R}^4$.

$$\begin{matrix}2&0&1&0\\2&0&0&1\\0&2&1&0\\0&2&0&1\end{matrix} \Rightarrow \begin{matrix}2&0&1&0\\0&0&-1&1\\0&2&1&0\\0&0&-1&1\end{matrix} \quad \begin{matrix}\\ 2 \leftarrow 2-1\\ \\ 4 \leftarrow 4-3\end{matrix}$$

$$\left.\begin{matrix}(2,0,1,0)\\(0,2,1,0)\\(0,0,-1,1)\end{matrix}\right\}\text{ basis for } V$$

$$\left.(0,0,0,1)\right\}\text{ extend to } \mathbb{R}^4$$

[2] By least squares, find the equation of the form $y = ax + b$ which best fits the data

$$(x_1, y_1) = (-1, 0), \quad (x_2, y_2) = (0, 0), \quad (x_3, y_3) = (1, 0), \quad (x_4, y_4) = (2, 1)$$

x	y
-1	0
0	0
1	0
2	1

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

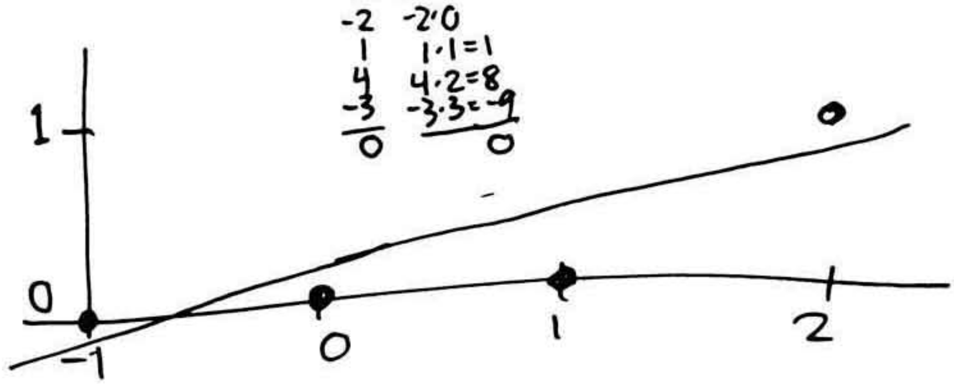
$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \Big/ 20 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \Big/ 20 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Big/ 10$$

x	y	(3x+1)/10	error		
-1	0	-2/10	-2/10	1	0
0	0	1/10	1/10	1	1
1	0	4/10	4/10	1	2
2	1	7/10	-3/10	1	3

$$\boxed{y = 3/10\, x + 1/10}$$

✓	✓
-2	-2·0
1	1·1=1
4	4·2=8
-3	-3·3=-9
0	0



[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the subspace V spanned by $(1,-1,0)$ and $(0,2,1)$. Find the matrix A which represents L in standard coordinates.

$$\left.\begin{matrix}(1,-1,0)\\(0,2,1)\end{matrix}\right\} \quad \begin{matrix}x+y-2z=0\\(1,1,-2)\cdot(x,y,z)=0\end{matrix}$$

$V \quad x+y-2z=0$ $\qquad$ A projects ortho to V

$W \quad (1,1,-2)$ $\qquad$ B " " " W

$A+B=I$

$$B = \begin{array}{c|ccc} & 1 & 1 & -2 \\ \hline 1 & 1 & 1 & -2 \\ 1 & 1 & 1 & -2 \\ -2 & -2 & -2 & 4 \end{array} \Big/ 6$$

$$A = I - B = \left(\begin{bmatrix} 6 & & \\ & 6 & \\ & & 6 \end{bmatrix} - \begin{pmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{pmatrix}\right) \Big/ 6$$

$$\boxed{A = \begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 6}$$

Check:

$$\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} \Big/ 6 \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ -6 & 12 & 0 \\ 0 & 6 & 0 \end{bmatrix} \Big/ 6 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\underbrace{\quad\quad}_{V} \; \underbrace{\quad}_{W}$$

[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the subspace V spanned by $(1,-1,0)$ and $(0,2,1)$. Find the matrix A which represents L in standard coordinates.

$$(1,-1,0) \Rightarrow (1,-1,0)$$

$$(0,2,1) \Rightarrow (0,2,1) - \frac{(0,2,1)\cdot(1,-1,0)}{(1,-1,0)\cdot(1,-1,0)}(1,-1,0)$$

$$= (0,2,1) - \left(\frac{-2}{2}\right)(1,-1,0)$$

$$= (1,1,1) \perp \text{ to } (1,-1,0)$$

project to $(1,-1,0)$:

$$\frac{(x,y,z)\cdot(1,-1,0)}{(1,-1,0)\cdot(1,-1,0)}(1,-1,0) = \begin{bmatrix}1\\-1\\0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\end{bmatrix}/2\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 0\end{bmatrix}/2\begin{bmatrix}x\\y\\z\end{bmatrix}$$

project to $(1,1,1)$:

$$\frac{(x,y,z)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = \begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix}/3\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}/3\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$(1,-1,0) \perp (1,1,1)$ so projections add to give projection to subspace

$$\begin{bmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 0\end{bmatrix}/2 + \begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}/3 = \begin{bmatrix}3 & -3 & 0\\-3 & 3 & 0\\0 & 0 & 0\end{bmatrix}/6 + \begin{bmatrix}2 & 2 & 2\\2 & 2 & 2\\2 & 2 & 2\end{bmatrix}/6$$

$$= \begin{bmatrix}5 & -1 & 2\\-1 & 5 & 2\\2 & 2 & 2\end{bmatrix}/6$$

[3] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which projects orthogonally onto the subspace V spanned by $(1,-1,0)$ and $(0,2,1)$. Find the matrix A which represents L in standard coordinates.

Find nearest solution for $\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}}_{}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \left(\begin{array}{c} Ax = b \\ \Downarrow \\ A^T A x = A^T b \end{array} \right)$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} / 6$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} / 6 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} / 6 \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}}_{\begin{bmatrix} 5 & -1 & 2 \\ 2 & 2 & 2 \end{bmatrix} / 6}}_{} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 5 & -1 & 2 \\ -1 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix} / 6}$$

[4] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(2) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

$$\{ax^2+bx+c\} = V = \{(a,b,c)\} \cong \mathbb{R}^3$$

$$f(2) = a4 + b2 + c = 0 \qquad (4,2,1)\cdot(a,b,c) = 0$$

W spanned by $(1,-2,0), (0,1,-2)$

$$x^2-2x \qquad x-2$$

$$V_1 = x-2 \Rightarrow w_1 = x-2$$

$$V_2 = x^2-2x \Rightarrow w_2 = (x^2-2x) - \frac{\langle x^2-2x, x-2\rangle}{\langle x-2, x-2\rangle}(x-2)$$

$$\langle x^2-2x, x-2\rangle = \int_0^1 (x^3-2x^2-2x^2+4x)\,dx = \frac{1}{4} - \frac{4}{3} + \frac{4}{2} = \frac{11}{12}$$

$$\langle x-2, x-2\rangle = \int_0^1 (x^2-2x-2x+4)\,dx = \frac{1}{3} - \frac{4}{2} + 4 = \frac{7}{3}$$

$$\frac{11}{12}\Big/\frac{7}{3} = \frac{33}{12\cdot 7} = \frac{11}{28} \qquad w_2 = x^2 - \frac{67}{28}x + \frac{22}{28}$$

$$\boxed{\begin{array}{l} x-2 \\ x^2 - \frac{67}{28}x + \frac{22}{28} \end{array}}$$

check $\quad 2^2 - \frac{67}{28}2 + \frac{22}{28} = (4\cdot 28 - 2\cdot 67 + 22)/28$

$$= (112 - 134 + 122)/28 = 0 \quad ✓$$

$$\int_0^1 (x-2)\left(x^2 - \frac{67}{28}x + \frac{22}{28}\right)dx = \frac{1}{4} - \frac{67}{28}\frac{1}{3} + \frac{22}{28}\frac{1}{2} - \frac{2}{3} + \frac{67}{28} - 2\frac{22}{28} = 0 \quad ✓$$

[4] Let V be the vector space of all polynomials of degree $\leqslant 2$ in the variable x with coefficients in $\mathbb{R}$. Let W be the subspace of polynomials satisfying $f(2) = 0$. Find an orthogonal basis for W with respect to the inner product

$$\langle f, g \rangle \;=\; \int_0^1 f(x)g(x)\,dx$$

$$x-2, \quad (ax+b)(x-2)$$

want $\displaystyle\int_0^1 (x-2)^2(ax+b)\,dx = 0$

$$a\int_0^1 (x-2)^2 x\,dx + b\int_0^1 (x-2)^2\,dx = 0$$

$$a\int_0^1 [x^3-4x^2+4x]\,dx + b\int_0^1 [x^2-4x+4]\,dx = 0$$

$$a\left(\frac{1}{4}-\frac{4}{3}+\frac{4}{2}\right) + b\left(\frac{1}{3}-\frac{4}{2}+4\right) = 0$$

$\times 12$ $\quad a(3-16+24) + b(4-24+48) = 0$

$$11a + 28b = 0$$

$$a = 28, \quad b = -11$$

$$(28x-11)(x-2) = 28x^2 - 67x + 22$$

$$\boxed{x-2, \quad 28x^2-67x+22}$$

[5] Find an orthogonal basis for the subspace of $\mathbb{R}^4$ defined by the equation $w + x - 2y - 2z = 0$. Extend this basis to a orthogonal basis for $\mathbb{R}^4$.

$(1,-1,0,0)$
$(0,0,1,-1)$

need third vector $\perp$ to these and $(1,1,-2,-2)$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(s,s,t,t) $\quad (1,1,-2,-2)\cdot(s,s,t,t) = 2s-4t = 0$

$(2,2,1,1)$ $\quad s=2, t=1$

$$\boxed{\begin{array}{l} \left.\begin{array}{l}(1,-1,0,0)\\(0,0,1,-1)\\(2,2,1,1)\end{array}\right\} \perp \text{ basis for } w+x-2y-2z=0 \\ \left.(1,1,-2,-2)\right\} \text{ extend to } \mathbb{R}^4 \end{array}}$$