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Exam 3 (Final)

Linear Algebra, Dave Bayer, December 20, 2007

Name: Answer Key

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	[7] (5 pts)	[8] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation.

[1] Find an orthogonal basis for the subspace V of $\mathbb{R}^4$ spanned by the rows of the matrix

$$\begin{array}{l} v_1 = \\ v_2 = \\ v_3 = \end{array} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

This is hyperplane given by $(w,x,y,z)\cdot(1,-1,1,-1)=0$

Rank 3, expect 3 vectors.

Rows of matrix are independent, take as v_1, v_2, v_3.

$$w_1 = v_1 = (1,1,0,0)$$

$$w_2 = v_2 - \left(\frac{v_2\cdot w_1}{w_1\cdot w_1}\right) w_1 = (0,2,2,0) - \frac{2}{2}(1,1,0,0) = (-1,1,2,0)$$

$$w_3 = v_3 - \left(\frac{v_3\cdot w_1}{w_1\cdot w_1}\right) w_1 - \left(\frac{v_3\cdot w_2}{w_2\cdot w_2}\right) w_2$$

$$= (0,0,3,3) - \frac{0}{\sim}\sim - \frac{6}{6}(-1,1,2,0) = (1,-1,1,3)$$

$$\boxed{\left\{\begin{matrix}(1,1,0,0)\\(-1,1,2,0)\\(1,-1,1,3)\end{matrix}\right\}}$$

checks $\perp$ ✓

$w-x+y-z=0$ ✓

[2] By least squares, find the equation of the form $z = ax + by + c$ which best fits the data

$$(x_1, y_1, z_1) = (0,0,1), \quad (x_2, y_2, z_2) = (1,0,1), \quad (x_3, y_3, z_3) = (0,1,0), \quad (x_4, y_4, z_4) = (1,1,2)$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 4 & 4 \end{array}\right] \underset{\div 2}{\rightleftarrows} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 3 \\ 1 & 1 & 2 & 2 \end{array}\right] \begin{matrix} \\ -2① \\ -① \end{matrix} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & -3 & -2 & -1 \\ 0 & -1 & 0 & 0 \end{array}\right]$$

$$\begin{matrix} +2③ \\ -3③ \\ *(-1) \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 \end{array}\right] \begin{matrix} +② \\ \div(-2) \\ \\ \end{matrix} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \end{array}\right] \rightleftarrows \left[\begin{array}{c|c} I & \begin{matrix} 1 \\ 0 \\ \frac{1}{2} \end{matrix} \end{array}\right]$$

$$\boxed{z = 1x + 0y + \tfrac{1}{2}}$$

$a = 1$
$b = 0$
$c = \frac{1}{2}$

check: $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$ ✓

[3] Let V be the subspace of $\mathbb{R}^4$ spanned by the rows of the matrix

$$\text{rank } 2 \left\{ \begin{array}{cl} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{bmatrix} & \begin{array}{l} v_1 \\ v_2 \\ v_1+v_2 \text{, dependent} \end{array} \end{array} \right.$$

Find the matrix A which projects $\mathbb{R}^4$ orthogonally onto the subspace V.

V has orthogonal basis $\quad w_1 = v_1, \quad w_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1$

$$\underbrace{\{(1,1,1,0)}_{w_1}, \underbrace{(-2,1,1,3)\}}_{w_2} \qquad = (1,1,1,0) \qquad = (0,1,1,1) - \tfrac{2}{3}(1,1,1,0)$$
$$= (-2/3, 1/3, 1/3, 1)$$
$$\sim (-2,1,1,3)$$

$$A(1,0,0,0) = \left(\frac{(1,0,0,0)\cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{(1,0,0,0)\cdot w_2}{w_2 \cdot w_2}\right) w_2$$
$$= \tfrac{1}{3}(1,1,1,0) - \tfrac{2}{15}(-2,1,1,3) = (9,3,3,-6)/15$$
$$= (3,1,1,-2)/5$$

$$A(0,1,0,0) = \left(\frac{(0,1,0,0)\cdot w_1}{w_1 \cdot w_1}\right) w_1 + \left(\frac{(0,1,0,0)\cdot w_2}{w_2 \cdot w_2}\right) w_2$$
$$= \tfrac{1}{3}(1,1,1,0) + \tfrac{1}{15}(-2,1,1,3) = (3,6,6,3)/15$$
$$= (1,2,2,1)/15$$

$A(0,0,1,0) = A(0,1,0,0)$ flipped $\}$ by symmetry.
$A(0,0,0,1) = A(1,0,0,0)$ flipped

so $\boxed{A = \begin{bmatrix} 3 & 1 & 1 & -2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -2 & 1 & 1 & 3 \end{bmatrix} / 5}$

check: $\begin{bmatrix} 3 & 1 & 1 & -2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -2 & 1 & 1 & 3 \end{bmatrix} / 5 \; \begin{array}{c} \begin{array}{cc} \overbrace{\quad}^{V} & \overbrace{\quad}^{\perp V} \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & -1 & +1 \\ 1 & 1 & -1 & +1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

[4] Let V be the vector space of all polynomials $f(x)$ of degree $\leqslant 3$. Find a basis for the subspace W defined by

$$f(0) = f(1) = f(2)$$ 2 conditions

Extend this basis to a basis for V.

$$f(x) = ax^3 + bx^2 + cx + d$$
$$f(0) = d$$
$$f(1) = a + b + c + d$$
$$f(2) = 8a + 4b + 2c + d$$

$$f(1) - f(0) = a + b + c = 0$$
$$f(2) - f(1) = 7a + 3b + c = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 7 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

$$\boxed{W = \{ x^3 - 3x^2 + 2x,\ 1 \}}$$

$x=0$: 0 ✓

$x=1$: 0

$x=2$: 0 ✓

extend to basis of V

$$\boxed{\{ x^3 - 3x^2 + 2x,\ 1,\ x^2,\ x \}}$$

[5] Define the inner product of two polynomials f and g by the rule

$$\langle f, g\rangle \quad = \quad \int_0^1 f(x)\, g(1-x)\, dx$$

Using this definition of the inner product, find an orthogonal basis for the vector space of all polynomials of degree $\leqslant 2$.

$$\langle ax^2+bx+c,\ rx^2+sx+t\rangle = [a\ b\ c]\ M \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

where $M = \begin{bmatrix} \langle x^2,x^2\rangle & \langle x^2,x\rangle & \langle x^2,1\rangle \\ \langle x,x^2\rangle & \langle x,x\rangle & \langle x,1\rangle \\ \langle 1,x^2\rangle & \langle 1,x\rangle & \langle 1,1\rangle \end{bmatrix}$

Note that $\langle g,f\rangle = \int_{x=0}^{x=1} g(x)f(1-x)\,dx = -\int_{y=1}^{y=0} g(1-y)f(y)\,dy = \int_0^1 f(x)g(1-x)\,dx$

$y=1-x$
$dy=-dx$

$= \langle f,g\rangle$

$\langle 1,1\rangle = \int_0^1 1\,dx = 1$

$\langle x,1\rangle = \int_0^1 x\,dx = 1/2$

$\langle x^2,1\rangle = \int_0^1 x^2 dx = 1/3$

$\langle x,x\rangle = \int_0^1 x(1-x)\,dx = \frac{1}{2}-\frac{1}{3} = 1/6$

$\langle x^2,x\rangle = \int_0^1 x^2(1-x)\,dx = \frac{1}{3}-1/4 = 1/12$

$\langle x^2,x^2\rangle = \int_0^1 x^2(1-2x+x^2)\,dx = \frac{1}{3}-2\frac{1}{4}+\frac{1}{5} = \frac{10-15+6}{30} = \frac{1}{30}$

so $M = \begin{bmatrix} 1/30 & 1/12 & 1/3 \\ 1/12 & 1/6 & 1/2 \\ 1/3 & 1/2 & 1 \end{bmatrix}$

$\{v_1, v_2, v_3\} = \{1, x, x^2\}$

$w_1 = v_1 = 1$

$w_2 = v_2 - \frac{\langle v_2,w_1\rangle}{\langle w_1,w_1\rangle} w_1 = x - \frac{\langle x,1\rangle}{\langle 1,1\rangle} 1 = x - 1/2$

$w_3 = v_3 - \frac{\langle v_3,w_1\rangle}{\langle w_1,w_1\rangle} w_1 - \frac{\langle v_3,w_2\rangle}{\langle w_2,w_2\rangle} w_2 = x^2 - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle} 1 - \frac{\langle x^2, x-\frac{1}{2}\rangle}{\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle}(x-\frac{1}{2})$

$= x^2 - \frac{1}{3}(1) - (x-1/2)$

$=$

$x^2 - x + 1/6$

$\langle x^2, x-\frac{1}{2}\rangle = [1\,0\,0]\begin{bmatrix} M \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = -1/12$

$\langle x-\frac{1}{2}, x-\frac{1}{2}\rangle = [0\ 1\ {-\frac{1}{2}}]\begin{bmatrix} M \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{bmatrix} = \frac{1}{6}-\frac{1}{4}-\frac{1}{4}+\frac{1}{4} = \frac{-1}{12}$

$\Rightarrow$ next page

5, continued

Check: $\langle x-\frac{1}{2}, 1\rangle = [0 \; 1 \; -\frac{1}{2}]\begin{bmatrix} & M & \end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$

$\langle x^2-x+\frac{1}{6}, x-\frac{1}{2}\rangle = [1 \; -1 \; \frac{1}{6}]\begin{bmatrix} & M & \end{bmatrix}\begin{bmatrix}0\\1\\-\frac{1}{2}\end{bmatrix} = 0$

$\cancel{\langle x^2-x+\frac{1}{6}, x^2-x+\frac{1}{6}\rangle = [1 \; -1 \; \frac{1}{6}]\begin{bmatrix} & M & \end{bmatrix}\begin{bmatrix}1\\-1\\\frac{1}{6}\end{bmatrix}}$

$\langle x^2-x+\frac{1}{6}, 1\rangle = [1 \; -1 \; \frac{1}{6}]\begin{bmatrix} & m & \end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = 0$

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othogonal basis = $\boxed{\left\{ 1,\ x-\frac{1}{2},\ x^2-x+\frac{1}{6} \right\}}$

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if one starts with $\{x^2, x, 1\}$ (reversed order for basis)

one gets $\{x^2,\ x-\frac{5}{2}x^2,\ 1-8x+10x^2\}$

[6] Express the following quadratic form as a linear combination of squares of orthogonal linear forms:

$$3x^2 + 4xy + 6y^2$$

$$3x^2+4xy+6y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \qquad \begin{aligned} \lambda_1 + \lambda_2 &= \operatorname{trace}(A) = 9 \\ \lambda_1 \lambda_2 &= \det(A) = 18 - 4 = 14 \\ \lambda &= 2, 7 \end{aligned}$$

$$A - 2I: \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 0$$

$$A - 7I: \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$$3x^2+4xy+6y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \Big/ 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} 2x-y & x+2y \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \Big/ 5 \begin{bmatrix} 2x-y \\ x+2y \end{bmatrix}$$

$$= \boxed{2/5\,(2x-y)^2 + 7/5\,(x+2y)^2}$$

checks: $(2,-1)\cdot(1,2) = 0$, so forms are $\perp$

$$\begin{aligned} 2(2x-y)^2 + 7(x+2y)^2 &= 2(4x^2 - 4xy + y^2) \\ &\quad + 7(x^2 + 4xy + 4y^2) \\ &\quad \overline{15x^2 + 20xy + 30y^2} \\ &= 5(3x^2 + 4xy + 6y^2) \end{aligned}$$

[7] Express the following quadratic form as a linear combination of squares of orthogonal linear forms:

$$2xy + 4xz + 4yz + 3z^2$$

$$2xy+4xz+4yz+3z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix} \qquad \lambda_1+\lambda_2+\lambda_3 = 3$$

$$\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 = \underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{-1} + \underbrace{\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}}_{-4} + \underbrace{\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}}_{-4} = -9$$

$$\lambda_1\lambda_2\lambda_3 = 0\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} - 1\underbrace{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}}_{-1} + 2\underbrace{\begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}}_{2} = 5$$

so $\lambda = -1, -1, 5$

$$A-(-1)I: \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = 0 \quad \text{(choose } \perp\text{)}$$

$$A-5I: \begin{bmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = 0 \quad \text{(take cross product of eigenvectors for } -1\text{)}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ 2 & 2 & -2 \\ 1 & 1 & 2 \end{bmatrix} \Big/ 6 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{array}{|ccccc|} 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 \end{array} \quad \text{(for inverse)} \ \checkmark$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & & \\ & -1 & \\ & & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} 3 & & \\ & 2 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x-y & x+y-z & x+y+2z \end{bmatrix} \begin{bmatrix} -3 & & \\ & -2 & \\ & & 5 \end{bmatrix} \Big/ 6 \begin{bmatrix} x-y \\ x+y-z \\ x+y+2z \end{bmatrix}$$

$$= \boxed{-\tfrac{1}{2}(x-y)^2 - \tfrac{1}{3}(x+y-z)^2 + \tfrac{5}{6}(x+y+2z)^2}$$

$\Rightarrow$ (next page)

(7, continued)

checks:

$$\begin{array}{l|ccc} x-y & 1 & -1 & 0 \\ x+y-z & 1 & 1 & -1 \\ x+y+2z & 1 & 1 & 2 \end{array} \quad \begin{matrix} \perp ✓ \\ \perp ✓ \end{matrix} \quad \perp ✓$$

orthogonal forms

$$\begin{aligned}
&-3\,(x^2-2xy+y^2 \qquad\qquad\qquad\quad\;) \\
&-2\,(x^2+2xy+y^2-2xz-2yz+z^2) \\
&+5\,(x^2+2xy+y^2+4xz+4yz+4z^2) \\
\hline
&\quad 0+12xy+0+24xz+24yz+18z^2 \\
&= 6\,(2xy+4xz+4yz+3z^2) \quad ✓
\end{aligned}$$

[8] Find e^{At} for the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 \\ -2 & -1 & -1 \\ -8 & -4 & -1 \end{bmatrix}$$

$$\lambda_1+\lambda_2+\lambda_3 = 4-1-1 = 2$$

$$\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 = \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix} + \begin{vmatrix} 4 & 1 \\ -8 & -1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ -4 & -1 \end{vmatrix} = \quad 4-3 = \quad 1$$

$$\lambda_1\lambda_2\lambda_3 = 4\begin{vmatrix} -1 & -1 \\ -4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 1 \\ -4 & -1 \end{vmatrix} + (-8)\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix}$$

$$= 4(-3) + 2(2) - 8(-1) = 0$$

$$\Rightarrow \lambda = 0, 1, 1$$

$$A - 0I: \begin{bmatrix} 4 & 2 & 1 \\ -2 & -1 & -1 \\ -8 & -4 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = 0$$

$$B = A - 1I: \begin{bmatrix} 3 & 2 & 1 \\ -2 & -2 & -1 \\ -8 & -4 & -2 \end{bmatrix} \qquad B^2 = \begin{bmatrix} -3 & -2 & -1 \\ 6 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 0 \text{ but } B\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 0,$$

can't use

$$'' \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = 0$$

good.

$$B\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \quad B\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = 0$$

SO:

$$\underset{v_1}{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}} \xrightarrow{A} 0, \quad \underset{v_3}{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}} \xrightarrow{A-I} \underset{v_2}{\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}} \xrightarrow{A-I} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $Av_1 = 0$

$Av_2 = v_2$

$Av_3 = v_3 + v_2$

$\Rightarrow$ (next page)

8, continued

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} +3 & +2 & +1 \\ -6 & -3 & -2 \\ +4 & +2 & +1 \end{bmatrix} \Big/ 1$$

$$\begin{array}{c|cccc} -1 & 2 & 0 & -1 & 2 \\ \hline 0 & 1 & -2 & 0 & 1 \\ 1 & 0 & -3 & 1 & 0 \\ -1 & 2 & 0 & -1 & 2 \\ 0 & 1 & -2 & 0 & 1 \end{array}$$

$$e^{At} = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -6 & -3 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -6 & -3 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$+ e^t \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -6 & -3 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$+ te^t \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ -6 & -3 & -2 \\ 4 & 2 & 1 \end{bmatrix}$$

$$\begin{array}{c|ccc} & 3 & 2 & 1 \\ \hline -1 & -3 & -2 & -1 \\ 2 & 6 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array} \qquad \underbrace{\begin{array}{c|ccc} & -6 & -3 & -2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & -6 & -3 & -2 \\ -2 & 12 & 6 & 4 \end{array} \quad \begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 1 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 \\ -3 & -12 & -6 & -3 \end{array}} \qquad \begin{array}{c|ccc} & 4 & 2 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 1 \\ -2 & -8 & -4 & -2 \end{array}$$

$$\boxed{e^{At} = \begin{bmatrix} -3 & -2 & -1 \\ 6 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} + e^t \begin{bmatrix} 4 & 2 & 1 \\ -6 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 0 & 0 & 0 \\ 4 & 2 & 1 \\ -8 & -4 & -2 \end{bmatrix}}$$