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Exam 2

Linear Algebra, Dave Bayer, November 8, 2007

Name: Answer Key

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	[6] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation.

[1] Use Cramer's rule to solve for x in the system of equations

$$\begin{bmatrix} a & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}}{\begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix}}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\boxed{\text{or } 1\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 3 - 3 + 1 = 1}$$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \underset{a\cdot 3}{a\begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix}} \underset{-3}{- 1\begin{vmatrix} 1 & 1 \\ 3 & 6 \end{vmatrix}} \underset{+1}{+ 1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = 3a-2$$

$$\Rightarrow \underbrace{\boxed{x = \frac{1}{3a-2}}}_{\text{answer}}$$

check:

col ops

$$y = \frac{\begin{vmatrix} a & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 6 \end{vmatrix}}{3a-2}$$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} a-1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 5 \end{vmatrix} = \begin{vmatrix} a-1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = (a-1)\cdot 3$$

$$\boxed{\text{or } a\begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 6 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 3a - 5 + 2 = 3a-3}$$

$$y = 3a-\cancel{3}3 / 3a-2$$

col ops

$$z = \frac{\begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix}}{3a-2}$$

$$\begin{vmatrix} a & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} a-1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} a-1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{vmatrix} = -a+1$$

$$z = -a+1 / 3a-2$$

$$\begin{bmatrix} a & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3a-3 \\ -a+1 \end{bmatrix} / 3a-2 = \begin{bmatrix} 3a-2 \\ 3a-2 \\ 3a-2 \end{bmatrix} / 3a-2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

[2] Find a basis for the subspace V of $\mathbb{R}^5$ defined by the following system of equations. Extend this basis to a basis for all of $\mathbb{R}^5$.

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 & 6 \\ \cancel{0} & \cancel{1} & \cancel{0} & \cancel{3} & \cancel{6} \\ 0 & 0 & 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|cc} \overbrace{1 \quad 0 \quad 0}^{\text{pivot}} & & & \overbrace{\boxed{2} \quad 5}^{\text{free}} & \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right] \qquad \begin{bmatrix} -2 & -5 \\ -3 & -6 \\ -4 & -7 \\ \hline 1 & 0 \\ 0 & 1 \end{bmatrix}$$

basis for V: (1st two rows)

extension to $\mathbb{R}^5$

$$\left[\begin{array}{ccccc} -2 & -3 & -4 & 1 & 0 \\ -5 & -6 & -7 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

read cols R to L

This is basis because lead ones in different cols, if we look from right

[3] Find A^n for the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$$

$$\left.\begin{array}{l} \text{trace}(A) = 2+5 = 7 = \lambda_1 + \lambda_2 \\ \det(A) = 10-4 = 6 = \lambda_1 \lambda_2 \end{array}\right\} \Longrightarrow \lambda = 1, 6$$

alternatively, $|A - \lambda I| = \lambda^2 - 7\lambda + 6 = (\lambda-1)(\lambda-6) = 0$

$A - 1I$: $\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

$A - 6I$: $\begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 0$

eigenbasis $= \{(1,-1), (1,4)\} = T$

usual basis $= S$

$$\underset{S \leftarrow S}{\overset{L}{\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}}} = \underset{S \leftarrow T}{\overset{I}{\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}}} \underset{T \leftarrow T}{\overset{L}{\begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}}} \underset{T \leftarrow S}{\overset{I}{\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}}} \Big/ 5$$

so $A^n = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & 6^n \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \Big/ 5$

$$\begin{array}{c|cc} & 4 & -1 \\ \hline 1 & 4 & -1 \\ -1 & -4 & 1 \end{array} \qquad \begin{array}{c|cc} & 1 & 1 \\ \hline 1 & 1 & 1 \\ 4 & 4 & 4 \end{array}$$

$$\boxed{A^n = 1^n \begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix} \Big/ 5 + 6^n \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \Big/ 5}$$

checks: $n=0$ $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \Big/ 5 = I$ ✓

$n=1$ $\begin{bmatrix} 10 & 5 \\ 20 & 25 \end{bmatrix} \Big/ 5 = A$ ✓

$n=2$ $\begin{bmatrix} 40 & 35 \\ 140 & 145 \end{bmatrix} \Big/ 5 = \begin{bmatrix} 8 & 7 \\ 28 & 29 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}^2 = \begin{bmatrix} 8 & 7 \\ 28 & 29 \end{bmatrix}$ ✓

[4] Find e^{At} for the matrix

$$A = \begin{bmatrix} 4 & -4 \\ 1 & 8 \end{bmatrix}$$

$$\text{trace}(A) = 12 = \lambda_1 + \lambda_2 \qquad \lambda = 6, 6$$
$$\det(A) = 36 = \lambda_1 \lambda_2$$

$$B = A - 6I = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

want

$$\begin{array}{ccccc} v_2 & \overset{B}{\longmapsto} & v_1 & \overset{B}{\longmapsto} & 0 \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \begin{bmatrix} -2 \\ 1 \end{bmatrix} & & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

$$A = 6I + B$$
$$Av_1 = 6v_1 \qquad = (6,0)^T$$
$$Av_2 = 6v_2 + v_1 = (1,6)^T$$

$$A = \begin{bmatrix} 4 & -4 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$
$$S \leftarrow S \qquad S \leftarrow T \qquad T \leftarrow T \qquad T \leftarrow S$$

$$e^{At} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{6t} & te^{6t} \\ 0 & e^{6t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{array}{c|cc} & 1 & 2 \\ \hline -2 & -2 & -4 \\ 1 & 1 & 2 \end{array}$$

$$A = 6I + B$$
$$e^{At} = e^{6t} I + te^{6t} B$$
$$\boxed{e^{At} = e^{6t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^{6t} \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}}$$

check: $t=0$: $I = 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$

$$\left(\frac{d}{dt} e^{At}\right)\Big|_{t=0} : \quad A = 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

[5] Find e^{At} for the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$\mathrm{trace}(A) = 2 = \lambda_1 + \lambda_2$
$\det(A) = 2 = \lambda_1 \lambda_2$ (?!)

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$A-(1+i)I$: $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = 0$

$A-(1-i)I$: $\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 0$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \underset{S \leftarrow S}{\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}} \begin{bmatrix} 1-i & 0 \\ 0 & 1+i \end{bmatrix} \underset{S \leftarrow T}{\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}} \Big/ 2$$

$$\begin{array}{c|cc} & 1 & -i \\ \hline 1 & 1 & -i \\ i & i & 1 \end{array} \qquad \begin{array}{c|cc} & -i & 1 \\ \hline i & 1 & i \\ 1 & -i & 1 \end{array}$$

$$e^{At} = e^{(1-i)t} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \Big/ 2 + e^{(1+i)t} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \Big/ 2$$

$e^{(1-i)t} = e^t e^{-it} = e^t \cos t - i e^t \sin t$
$e^{(1+i)t} = e^t e^{it} = e^t \cos t + i e^t \sin t$

$$e^{At} = e^t \cos t \left(\underbrace{\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \Big/ 2 + \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \Big/ 2}\right) + e^t \sin t \left(\underbrace{-i \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \Big/ 2 + i \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \Big/ 2} \right)$$

$$\boxed{e^{At} = e^t \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t \sin t \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$$

this makes sense: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts on $\mathbb{R}^2$ like 1 on $\mathbb{C}$
$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ " " " i on $\mathbb{C}$
$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ " " " $1+i$ on $\mathbb{C}$

so $e^{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} t}$ acts on $\mathbb{R}^2$ like $e^{(1+i)t}$ on $\mathbb{C}$

$$e^{(1+i)t} = \underset{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\underset{\downarrow}{1}} e^t \cos t + \underset{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}{\underset{\downarrow}{i}} e^t \sin t$$

[6] Find e^{At} for the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\lambda^3 - (1+1+1)\lambda^2 + \left(\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right)\lambda - 0 = 0$$

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0 \qquad \lambda(\lambda-1)(\lambda-2) = 0$$

$$\lambda = 0, 1, 2$$

$$A - 0\lambda: \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 0$$

$$A - 1\lambda: \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ +1 \\ 1 \end{bmatrix} = 0$$

$$A - 2\lambda: \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{S \leftarrow S} = \underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{S \leftarrow T} \underbrace{\begin{bmatrix} 0 & & \\ & 1 & \\ & & 2 \end{bmatrix}}_{T \leftarrow T} \underbrace{\begin{bmatrix} 0 & -1 & +1 \\ -1 & -1 & +1 \\ +1 & +1 & 0 \end{bmatrix}}_{T \leftarrow S} / +1 \quad ✓$$

$$\begin{array}{c|cccc} & 1 & -1 & 0 & 1 \;\; -1 \\ \hline -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{array}$$

inverse (then negate)

$$A = 0 \begin{array}{c} \\ 1 \\ -1 \\ 0 \end{array}\!\!\begin{array}{c} 0 \quad -1 \quad 1 \\ \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} + 1 \begin{array}{c} \\ -1 \\ 1 \\ 1 \end{array}\!\!\begin{array}{c} -1 \quad -1 \quad 1 \\ \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} + 2 \begin{array}{c} \\ 0 \\ 1 \\ 1 \end{array}\!\!\begin{array}{c} 1 \quad 1 \quad 0 \\ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

check ✓

$$I = 1\,(\;\text{〃}\;) + 1\,(\;\text{〃}\;) + 1\,(\;\text{〃}\;)$$

check ✓

$$\boxed{e^{At} = 1 \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} + e^{t} \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}}$$

$$e^{At}\Big|_{t=0} = I \quad ✓$$

$$\left(\frac{d}{dt} e^{At}\right)\Big|_{t=0} = A \quad ✓$$