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Exam 1

Linear Algebra, Dave Bayer, October 2, 2007

Name: Answer Key

[1] (5 pts)	[2] (5 pts)	[3] (5 pts)	[4] (5 pts)	[5] (5 pts)	TOTAL

Please work only one problem per page, starting with the pages provided. Clearly label your answer. If a problem continues on a new page, clearly state this fact on both the old and the new pages.

Do not use calculators or decimal notation.

[1] Use Gaussian elimination to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 0 & 1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \times(-1)$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & 3 & -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] +2(1)$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \\ 0 & 3 & -2 & 1 & 2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 & 2 & 0 \end{array}\right] \times(-1)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & -1 & -2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 3 & -2 & 1 & 2 & 0 \end{array}\right] +2(2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & -1 & -2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 4 & 1 & 2 & 3 \end{array}\right] -3(2)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & -1 & -2 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] \times\frac{1}{4}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & -2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] +3(3)$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right] +2(3)$$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} / 4}$$

check:

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} / 4 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} / 4$$

✓

[2] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 & 1 & 10 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 16 \end{bmatrix}$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & 8 \\ 0 & 1 & 1 & 0 & 2 & 1 & 10 & 16 \end{array}\right]$$

$$\left[\begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 9 \end{array}\right] - ①$$

FREE COLS

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix} \left[\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 0 \\ 7 & 0 & -1 & -2 & -4 \\ 0 & 0 & 1 & 0 & 0 \\ 8 & 0 & 0 & -3 & -5 \\ 0 & 0 & 0 & 1 & 0 \\ 9 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} 7 & \\ 8 & \ldots 0 \ldots \\ 9 & \end{array}\right]$$

particular homogeneous

$$\text{solutions} = \begin{bmatrix} 0 \\ 7 \\ 0 \\ 8 \\ 0 \\ 9 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ -4 \\ 0 \\ -5 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

[3] Express A as a product of elementary matrices, where

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0&2&0\\0&0&3\\1&2&3 \end{bmatrix} \xrightarrow{2 \rightleftarrows 3} \begin{bmatrix} 0&2&0\\1&2&3\\0&0&3 \end{bmatrix} \xrightarrow{1 \rightleftarrows 2} \begin{bmatrix} 1&2&3\\0&2&0\\0&0&3 \end{bmatrix} \xrightarrow{2 = \frac{1}{2} 2} \begin{bmatrix} 1&2&3\\0&1&0\\0&0&3 \end{bmatrix} \xrightarrow{3 = \frac{1}{3} 3} \begin{bmatrix} 1&2&3\\0&1&0\\0&0&1 \end{bmatrix} \xrightarrow{1 \leftarrow 1 - 2 \cdot 2} \begin{bmatrix} 1&0&3\\0&1&0\\0&0&1 \end{bmatrix} \xrightarrow{1 \leftarrow 1 - 3 \cdot 3} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$$

$$2 \rightleftarrows 3 \qquad 1 \rightleftarrows 2 \qquad 2 = 2 \cdot 2 \qquad 3 = 3 \cdot 3 \qquad 1 = 1 + 2 \cdot 2 \qquad 1 = 1 + 3 \cdot 3$$

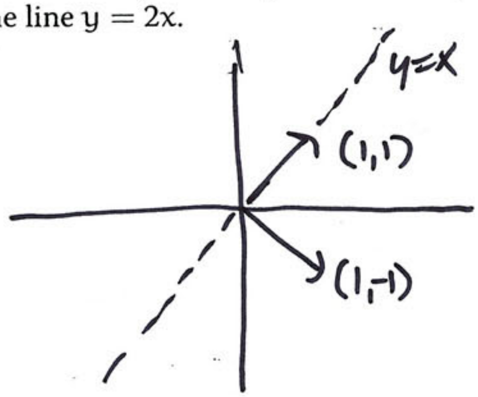
$$\boxed{A = \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix} \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&2&0\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0\\0&1&0\\0&0&3 \end{bmatrix} \begin{bmatrix} 1&2&0\\0&1&0\\0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&3\\0&1&0\\0&0&1 \end{bmatrix}}$$

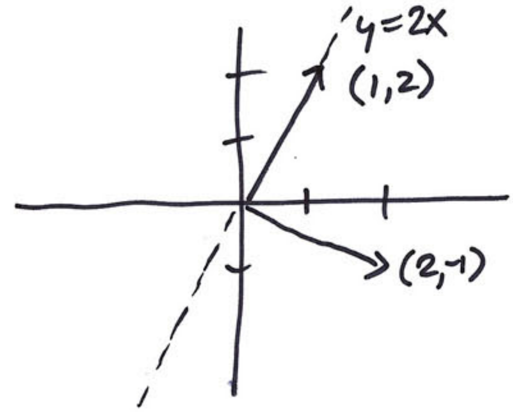
check:

$$\underbrace{\begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} \quad \underbrace{\begin{bmatrix} 1&0&0\\0&2&0\\0&0&3 \end{bmatrix} \quad \begin{bmatrix} 1&2&3\\0&1&0\\0&0&1 \end{bmatrix}}_{\begin{bmatrix} 1&2&3\\0&2&0\\0&0&3 \end{bmatrix}}}_{\begin{bmatrix} 0&2&0\\0&0&3\\1&2&3 \end{bmatrix}} \checkmark$$

check: only used <u>elementary matrices</u> (one step at a time) ✓

[4] Find a matrix representing the linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$ which reflects first across the line $y = x$, then across the line $y = 2x$.





B = reflect across $y=x$

$B\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$, $B\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$

combine:

$B\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$

right-multiply by $\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}^{-1}$:

$B = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}^{-1}$

$B = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}/2 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

check:

$\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$ ✓

C = reflect across $y=2x$

$C\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}-2\\1\end{bmatrix}$, $C\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}$

combine:

$C\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}-2 & 1\\1 & 2\end{bmatrix}$

right-multiply by $\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}^{-1}$:

$C = \begin{bmatrix}-2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}^{-1}$

$C = \begin{bmatrix}-2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix}/5 = \begin{bmatrix}-3 & 4\\4 & 3\end{bmatrix}/5$

check:

$\begin{bmatrix}-3 & 4\\4 & 3\end{bmatrix}/5\begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix} = \begin{bmatrix}-10 & 5\\5 & 10\end{bmatrix}/5$ ✓

$$A = \underbrace{C\,B}_{B \text{ then } C} = \begin{bmatrix}-3 & 4\\4 & 3\end{bmatrix}/5\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \boxed{\begin{bmatrix}4 & -3\\3 & 4\end{bmatrix}/5 = A}$$

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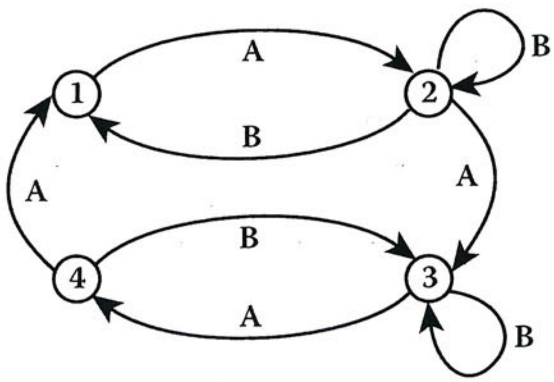
spot-checks: (2,1) reflects across $y=x$ to (1,2), fixed by reflection on $y=2x$
(-1,2) reflects " " " (2,-1), negated " " "

so $A\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix} = \begin{bmatrix}1 & -2\\2 & 1\end{bmatrix}$ (one could also use this to find A directly)

$\begin{bmatrix}4 & -3\\3 & 4\end{bmatrix}/5\begin{bmatrix}2 & -1\\1 & 2\end{bmatrix} = \begin{bmatrix}5 & -10\\10 & 5\end{bmatrix}/5$ ✓

[5] Compute a matrix giving the number of walks of length 4 between pairs of vertices of the following directed graph:



How many of these paths are labeled ABAB ?

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

both of form:

$$\begin{array}{c|c} & \text{from} \\ & ①②③④ \\ \hline \begin{matrix} ① \\ ② \\ ③ \\ ④ \end{matrix} & \boxed{\text{\#walks length 1}} \\ \text{to} & \end{array}$$

$$A+B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{square}} (A+B)^2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{square}} \boxed{(A+B)^4 = \begin{bmatrix} 3 & 5 & 3 & 2 \\ 3 & 6 & 2 & 3 \\ 5 & 10 & 6 & 5 \\ 2 & 5 & 3 & 3 \end{bmatrix}}$$

all walks, length 4

$$\overset{B}{\curvearrowleft}\ \overset{A}{\curvearrowleft}\ \overset{B}{\curvearrowleft}\ \overset{A}{\curvearrowleft}$$

$[B][A][B][A]$ gives similar table, just walks labeled ABAB

compute: $BA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad BABA = (BA)^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{5 walks in all, labeled ABAB}}$

find them as check:

1 to 1   1 to 2   1 to 3   2 to 3   3 to 3   as found by matrix