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Final Exam

Linear Algebra, Dave Bayer, December 21, 2006

Name: Solutions

[**1**] (5 pts)	[**2**] (5 pts)	[**3**] (5 pts)	[**4**] (5 pts)	[**5**] (5 pts)	**TOTAL**

Please work only one problem per page, starting with the pages provided, and identify all continuations clearly.

[**1**] Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D. Find the matrix e^{At}.

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answer:

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & \\ & 6 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} /5$$

$$e^{At} = e^{t} \begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix} /5 + e^{6t} \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} /5$$

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work:

$$\lambda^2 - (2+5)\lambda + \begin{vmatrix} 2 & 1 \\ 4 & 5 \end{vmatrix}$$
$$= \lambda^2 - 7\lambda + 6 = 0$$
$$(\lambda - 1)(\lambda - 6) = 0$$
$$\lambda_1 = 1 \quad \lambda_2 = 6$$

$$\lambda_1 = 1 \quad A - 1I: \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$
$$v_1 = (1, -1)$$
$$\lambda_2 = 6 \quad A - 6I: \begin{bmatrix} -4 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 0$$
$$v_2 = (1, 4)$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \\ & 6 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}}_{\begin{bmatrix} 4 & -1 \\ 6 & 6 \end{bmatrix}/5}}_{\begin{bmatrix} 10 & 5 \\ 20 & 25 \end{bmatrix}/5} /5$$

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$$① \quad \begin{array}{c|cc} & 4 & -1 \\ \hline 1 & 4 & -1 \\ -1 & -4 & 1 \end{array} \Rightarrow \begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix} /5$$

$$② \quad \begin{array}{c|cc} & 1 & 1 \\ \hline 1 & 1 & 1 \\ 4 & 4 & 4 \end{array} \Rightarrow \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} /5$$

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$$I = \begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix}/5 + \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}/5$$

$$A = 1\begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix}/5 + 6\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}/5 = \begin{bmatrix} 10 & 5 \\ 20 & 25 \end{bmatrix}/5$$

$$\Rightarrow \boxed{e^{At} = e^{t}\begin{bmatrix} 4 & -1 \\ -4 & 1 \end{bmatrix}/5 + e^{6t}\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}/5}$$

[2] Let $A = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D. Find the matrix e^{At}.

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answer:

$$A = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} + e^t \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$$

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work:

$$\lambda^2 - (-3+4)\lambda + \begin{vmatrix} -3 & -2 \\ 6 & 4 \end{vmatrix}$$
$$= \lambda^2 - \lambda = 0$$
$$\lambda(\lambda - 1) = 0$$
$$\lambda_1 = 0 \quad \lambda_2 = 1$$

$\lambda_1 = 0 \quad A - 0I: \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0$

$v_1 = (2, -3)$

$\lambda_2 = 1 \quad A - 1I: \begin{bmatrix} -4 & -2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$

$v_2 = (-1, 2)$

$$A = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}}_{\begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix} / 1} \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} / 1}_{\begin{bmatrix} 0 & 0 \\ 3 & 2 \end{bmatrix} / 1}$$

$$\begin{array}{c|cc} & 2 & 1 \\ \hline 2 & 4 & 2 \\ -3 & -6 & -3 \end{array} \Rightarrow_{1} \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix}$$

$$\begin{array}{c|cc} & 3 & 2 \\ \hline -1 & -3 & -2 \\ 2 & 6 & 4 \end{array} \Rightarrow_{2} \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$$

$$I = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} + \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$$

$$A = 0 \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} + 1 \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} + e^t \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}}$$

$(e^{0t} = 1)$

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[3] Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}$. Write A as CDC^{-1} for a diagonal matrix D. Find the matrix e^{At}.

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answer:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & & \\ & 1 & \\ & & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ -2 & -4 & 2 \\ 3 & 4 & 1 \end{bmatrix} \Big/ 4$$

$$e^{At} = e^{-t} \begin{bmatrix} +3 & 0 & -3 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \Big/ 4 + e^{t} \begin{bmatrix} -2 & -4 & 2 \\ 2 & 4 & -2 \\ -2 & -4 & 2 \end{bmatrix} \Big/ 4 + e^{3t} \begin{bmatrix} 3 & 4 & 1 \\ 0 & 0 & 0 \\ 3 & 4 & 1 \end{bmatrix} \Big/ 4$$

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work:

$$\lambda^3 - \underbrace{(1+1+1)}_{3} \lambda^2 + \left(\underbrace{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}}_{-1} + \underbrace{\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}}_{-3} + \underbrace{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}}_{3} \right) \lambda - \underbrace{\begin{vmatrix} 1 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{vmatrix}}_{} = 0$$

$$\lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

$$\begin{vmatrix} 1 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 3 \\ 1 & 1 & -1 \\ 0 & 0 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{vmatrix} = -3$$

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① ⇄ ②

find integer roots:

	-3	-1	1	3
3	3	3	3	3
$-\lambda$	3	1	-1	-3
$-3\lambda^2$	-27	-3	-3	-27
λ^3	-27	-1	1	27
		0	0	0

$$\lambda_1 = -1 \quad A-(-1)I: \begin{bmatrix} 2 & 2 & 2 \\ 1 & 2 & -1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} = 0 \qquad \begin{matrix} -6 \\ +4 \\ 2 \end{matrix}$$

$v_1 =$

$$\lambda_2 = 1 \quad A-1I: \begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$$

$v_2 =$

$$\lambda_3 = 3 \quad A-3I: \begin{bmatrix} -2 & 2 & 2 \\ 1 & -2 & -1 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \qquad \begin{matrix} 2 \\ 0 \\ 2 \end{matrix}$$

$v_3 =$

$$C = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \\ \downarrow & \downarrow & \downarrow \end{matrix} \\ \begin{bmatrix} -3 & 1 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

C^{-1}:

	-3	2	1	-3	2
1	-1	1	1	-1	
1	0	1	1	0	
-3	2	1	-3	2	
1	-1	1	1	-1	

$$C^{-1} = \begin{bmatrix} -1 & 0 & 1 \\ -2 & -4 & 2 \\ 3 & 4 & 1 \end{bmatrix} \Big/ 4$$

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$$A=\begin{bmatrix}1&2&2\\1&1&-1\\2&2&1\end{bmatrix}=\begin{bmatrix}-3&1&1\\2&-1&0\\1&1&1\end{bmatrix}\underbrace{\begin{bmatrix}-1&&\\&1&\\&&3\end{bmatrix}\begin{bmatrix}-1&0&1\\-2&-4&2\\3&4&1\end{bmatrix}/4}_{\begin{bmatrix}1&0&-1\\-2&-4&2\\9&12&3\end{bmatrix}/4}$$

$$✓\quad\begin{bmatrix}4&8&8\\4&4&-4\\8&8&4\end{bmatrix}/4$$

① ② ③

$$\begin{array}{c|ccc} & -1 & 0 & 1\\ \hline -3 & 3 & 0 & -3\\ 2 & -2 & 0 & 2\\ 1 & -1 & 0 & 1\end{array}\qquad\begin{array}{c|ccc} & -2 & -4 & 2\\ \hline 1 & -2 & -4 & 2\\ -1 & 2 & 4 & -2\\ 1 & -2 & -4 & 2\end{array}\qquad\begin{array}{c|ccc} & 3 & 4 & 1\\ \hline 1 & 3 & 4 & 1\\ 0 & 0 & 0 & 0\\ 1 & 3 & 4 & 1\end{array}$$

$$I=\begin{bmatrix}3&0&-3\\-2&0&2\\-1&0&1\end{bmatrix}/4+\begin{bmatrix}-2&-4&2\\2&4&-2\\-2&-4&2\end{bmatrix}/4+\begin{bmatrix}3&4&1\\0&0&0\\3&4&1\end{bmatrix}/4\quad ✓$$

$$A=-1\begin{bmatrix}3&0&-3\\-2&0&2\\-1&0&1\end{bmatrix}/4+1\begin{bmatrix}-2&-4&2\\2&4&-2\\-2&-4&2\end{bmatrix}/4+3\begin{bmatrix}3&4&1\\0&0&0\\3&4&1\end{bmatrix}/4\quad ✓$$

$$\boxed{e^{At}=e^{-t}\begin{bmatrix}3&0&-3\\-2&0&2\\-1&0&1\end{bmatrix}/4+e^{t}\begin{bmatrix}-2&-4&2\\2&4&-2\\-2&-4&2\end{bmatrix}/4+e^{3t}\begin{bmatrix}3&4&1\\0&0&0\\3&4&1\end{bmatrix}/4}$$

[4] Let $A = \begin{bmatrix} 8 & -1 \\ 1 & 6 \end{bmatrix}$. Find the matrix e^{At}.

answer:

$$e^{At} = e^{7t}\begin{bmatrix}1&0\\0&1\end{bmatrix} + te^{7t}\begin{bmatrix}1&-1\\1&-1\end{bmatrix}$$

work:

$$\lambda^2 - (8+6)\lambda + 49 = 0$$
$$\lambda^2 - 14\lambda + 49 = 0$$
$$(\lambda - 7)^2 = 0$$
$\lambda_1 = \lambda_2 = 7$ repeated root

$$\lambda = 7 \quad \overbrace{A - 7I}^{A_7} : \begin{bmatrix}1&-1\\1&-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 0$$

$v_1 = (1,1)$ no v_2, can't be diagonalized.

Can't find $v_1, v_2 \xrightarrow{A_7} 0$
so look for

$$\underset{(1,0)}{v_2} \xrightarrow{A_7} \underset{(1,1)}{v_1} \xrightarrow{A_7} 0$$

$$A = 7I + A_7$$
$$Av_1 = 7v_1 + A_7 v_1 = 7v_1$$
$$Av_2 = 7v_2 + A_7 v_2 = 7v_2 + v_1$$

so in V coords A looks like

$$\begin{bmatrix}7&1\\0&7\end{bmatrix}$$

$$A = \begin{bmatrix}8&-1\\1&6\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\underbrace{\begin{bmatrix}7&1\\0&7\end{bmatrix}\begin{bmatrix}0&1\\1&-1\end{bmatrix}}_{\begin{bmatrix}1&6\\7&-7\end{bmatrix}}$$

$$\begin{bmatrix}8&-1\\1&6\end{bmatrix}$$

$$e^{At} = \cancel{\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}7&1\\0&7\end{bmatrix}}$$

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}e^{7t}&te^{7t}\\0&e^{7t}\end{bmatrix}\begin{bmatrix}0&1\\1&-1\end{bmatrix}$$

$$\boxed{e^{At} = e^{7t}\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{I} + te^{7t}\underbrace{\begin{bmatrix}1&-1\\1&-1\end{bmatrix}}_{A_7}}$$

because $A = 7I + A_7$, and $A_7^2 = 0$

$$e^{At} = e^{7It}e^{A_7 t} = e^{7t}I(I + tA_7)$$

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[5] Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Find the matrix e^{At}.

answer:

$$e^{At} = e^t\cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t \sin t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

work:

$$\lambda^2 - (1+1)\lambda + \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 0$$

$$\lambda^2 - 2\lambda + 2 = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$(\lambda - (1-i))(\lambda - (1+i))$$

$$= \lambda^2 - [\underbrace{(1-i)+(1+i)}_{2}]\lambda + \underbrace{(1-i)(1+i)}_{2}$$

indeed, $\lambda_1 = 1-i$, $\lambda_2 = 1+i$

$$\lambda_1 = 1-i \quad \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = 0$$

$$v_1 = (1, -i)$$

$$\lambda_2 = 1+i$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = 0$$

$$v_2 = (1, i)$$

$$\begin{array}{cc} & \begin{array}{cc} i & -1 \end{array} \\ \begin{array}{c} 1 \\ -i \end{array} & \begin{array}{|cc} i & -1 \\ 1 & i \end{array} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} i & 1 \end{array} \\ \begin{array}{c} 1 \\ i \end{array} & \begin{array}{|cc} i & 1 \\ -1 & i \end{array} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} 1 & i \end{array} \\ \begin{array}{c} 1 \\ -i \end{array} & \begin{array}{|cc} 1 & i \\ -i & 1 \end{array} \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} 1 & -i \end{array} \\ \begin{array}{c} 1 \\ i \end{array} & \begin{array}{|cc} 1 & -i \\ i & 1 \end{array} \end{array}$$

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 1-i & \\ & 1+i \end{bmatrix}\begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}/2i$$

$$\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}/2 \qquad \begin{bmatrix} 1-i & 1+i \\ 1+i & 1-i \end{bmatrix}/2 \qquad \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}/2$$

$$I = \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}/2i + \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}/2i$$

$$= \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}/2 + \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}/2$$

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$$A = (1-i)\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}/2 + (1+i)\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}/2 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\boxed{e^{At} = e^{(1-i)t}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}/2 + e^{(1+i)t}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}/2}$$

$$= e^{t}(e^{-it})\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}/2 + e^{t}(e^{it})\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}/2$$

use Euler's formula: $e^{it} = \cos t + i\sin t$

$$= e^{t}(\cos t - i\sin t)\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}/2 + e^{t}(\cos t + i\sin t)\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}/2$$

$$= e^{t}\cos t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{t}\sin t\left(\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}/2 + \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}/2\right)$$

$$\boxed{e^{At} = e^{t}\cos t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{t}\sin t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$