sorting code

5	5	5

Exam 1

Linear Algebra, Dave Bayer, September 26, 2006

Name: Solutions

[**1**] (5 pts)	[**2**] (5 pts)	[**3**] (5 pts)	[**4**] (5 pts)	[**5**] (5 pts)	**TOTAL**

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[**1**] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right] \xrightarrow{③ = \frac{1}{3}③} \left[\begin{array}{ccc|c} 2 & 1 & 4 & 1 \\ 1 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[② = ② - 4③]{① = ① - 4③} \left[\begin{array}{ccc|c} 2 & 1 & 0 & -3 \\ 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{① \rightleftarrows ②} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 2 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{② = ② - 2①} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{② = -\frac{1}{3}②} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{① = ① - 2②} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

check:

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \checkmark$$

solution:

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$$

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[2] What is the set of all solutions to the following system of equations?

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 \end{array}\right]$$

reorder (3), (4), (1), (2)

$$\left[\begin{array}{cccc|c} -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & -1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right]$$

(1) = −(1)
(2) = −(2)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & -1 & 0 \end{array}\right]$$

(3) = (3) − (1)
(4) = (4) − (1)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -1 \end{array}\right]$$

(3) = (3) − (2)
(4) = (4) + (2)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underbrace{\quad\quad}_{\text{control cols}} \quad \underbrace{\quad\quad}_{\text{free cols}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left[\begin{array}{c|cc} 1 & -1 & 0 \\ 1 & 0 & -1 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

particular homog

Solution:

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}$$

check:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1-s \\ 1-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -1 \end{bmatrix} \checkmark$$

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[**3**] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 8 \end{bmatrix} \xrightarrow{① \rightleftarrows ②} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 1 & 8 \end{bmatrix} \xrightarrow{③ = ③ - 3①} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{③ = ③ - ②} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{② = ② - ③} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{① = ① - 2③} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$① \rightleftarrows ②$ $\quad$ $③ = ③ + 3①$ $\quad$ $③ = ③ + ②$ $\quad$ $② = ② + ③$ $\quad$ $① = ① + 2③$

$$\boxed{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{solution}}$$

check

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 1 & 8 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 1 & 8 \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 8 \end{bmatrix} \checkmark$$

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[4] What are the determinants of the following matrices? What is the determinant of the corresponding $n \times n$ matrix?

$$[3], \quad \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$F(1)$ $\quad F(2)$ $\quad F(3)$ $\quad F(4)$ $\quad F(5)$

$F(1)=3$
$F(2)=8$
$F(3)=21$
$F(4)=55$
$F(5)=144$

$$+3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = +3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 \end{vmatrix}$$

$$F(3) = 3F(2) - F(1) = 3\cdot 8 - 3 = 21$$

$$+3\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$$F(4) = 3F(3) - F(2) = 3\cdot 21 - 8 = 55$$

$$+3\begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} = 3\begin{vmatrix} 3 & 1 & & \\ 1 & 3 & 1 & \\ & 1 & 3 & 1 \\ & & 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix}$$

$$F(5) = 3F(4) - F(3)$$
$$= 3\cdot 55 - 21 = 144$$

solution:

n	det
2	8
3	21
4	55
5	144

pattern: $F(n) = 3F(n-1) - F(n-2)$
where $F(1)=3$, $F(2)=8$

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[5] What is the determinant of the following matrix? What is the determinant of the corresponding $n \times n$ matrix?

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 & 0 \\ 3 & 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

$2 = 2 - 1$
$3 = 3 - 1$
$4 = 4 - 1$
$5 = 5 - 1$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 \end{bmatrix}$$

$3 = 3 - 2$
$4 = 4 - 2$
$5 = 5 - 2$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$4 = 4 - 3$
$5 = 5 - 3$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$5 = 5 - 4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$\overbrace{4+3+2+1}^{10}$ swaps to sort

$(-1)^{10} = +1$

* adding one row to another has no effect on det
* swaps negate det, so 10 swaps have no effect
* det of triangular matrix is product of diagonal

$\Rightarrow$ $\boxed{\det = 1}$

$n \times n$ case same, except

n	swaps	det	n mod 4
2	1 = 1	−1	2
3	2+1 = 3	−1	3
4	3+2+1 = 6	+1	0
5	4+3+2+1 = 10	+1	1
6	5+10 = 15	−1	2
7	6+15 = 21	−1	3
8	7+21 = 28	+1	0
⋮			⋮

$$\boxed{\begin{array}{l} n \times n \det = 1 \quad \text{for } n \bmod 4 = 0, 1 \\ n \times n \det = -1 \quad \text{for } n \bmod 4 = 2, 3 \end{array}}$$

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(alternate solution)

[5] What is the determinant of the following matrix? What is the determinant of the corresponding $n \times n$ matrix?

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}}_{F(5)} \quad F(1) = 1,\ F(2) = -1,\ F(3),\ F(4)$$

$$+1\underbrace{\begin{vmatrix} 2&2&2&1 \\ 3&3&2&1 \\ 4&3&2&1 \\ 4&3&2&1 \end{vmatrix}}_{0} - 2\underbrace{\begin{vmatrix} 1&1&1&1 \\ 3&3&2&1 \\ 4&3&2&1 \\ 4&3&2&1 \end{vmatrix}}_{0} + 3\begin{vmatrix} 1&1&1&1 \\ 2&2&2&1 \\ 4&3&2&1 \\ 4&3&2&1 \end{vmatrix} - 4\underbrace{\begin{vmatrix} 1&1&1&1 \\ 2&2&2&1 \\ 3&3&2&1 \\ 4&3&2&1 \end{vmatrix}}_{F(4)} + 5\underbrace{\begin{vmatrix} 1&1&1&1 \\ 2&2&2&1 \\ 3&3&2&1 \\ 4&3&2&1 \end{vmatrix}}_{F(4)}$$

rows repeat, det = 0

pattern:

$$F(2) = 1F(1) - 2F(1) = -F(1) = -1$$
$$F(3) = -2F(2) + 3F(2) = F(2) = -1$$
$$F(4) = 3F(3) - 4F(3) = -F(3) = 1$$
$$F(5) = -4F(4) + 5F(4) = F(4) = 1$$
$$F(6) = \qquad = -F(5)$$

$\vdots$

$$\boxed{\begin{gathered} F(n) = (-1)^{n-1} F(n-1) = \begin{cases} -1, & n = 2,3 \bmod 4 \\ 1, & n = 0,1 \bmod 4 \end{cases} \\ \text{where } F(1) = 1 \\ F(5) = 1 \end{gathered}}$$

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