Exam 2

Linear Algebra, Dave Bayer, November 9, 2000

Name: Solutions

ID: ____________________ **School:** ____________________

[1] (6 pts)	[2] (6 pts)	[3] (6 pts)	[4] (6 pts)	[5] (6 pts)	TOTAL

Please work only one problem per page, starting with the pages provided, and number all continuations clearly. Only work which can be found in this way will be graded.

Please do not use calculators or decimal notation.

[1] Let

$$\mathbf{v}_1 = (1,-1,0,0), \quad \mathbf{v}_2 = (-1,1,0,0), \quad \mathbf{v}_3 = (1,-1,1,-1), \quad \mathbf{v}_4 = (-1,1,1,1).$$

Find a basis for the subspace $V \subset \mathbb{R}^4$ spanned by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{(2)=(2)+(1) \\ (3)=(3)-(1) \\ (4)=(4)+(1)}} \begin{array}{c} \times \\ \\ \times \\ \times \end{array} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

leading 1 alone in column, so this row part of basis

det $= 2 \neq 0$ so these rows are independent

insert vectors as rows of matrix (So V is rowspace)

now easier to understand (V is <u>still</u> rowspace)

Thus $\boxed{\{(1,-1,0,0),\ (0,0,1,-1),\ (0,0,1,1)\} \text{ is a basis for } V.}$

Checks:

basis as rows $\rightarrow$ $\begin{bmatrix} 1 & \cancel{-1} & 0 & 0 \\ 0 & \cancel{0} & 1 & -1 \\ 0 & \cancel{0} & 1 & 1 \end{bmatrix}$ det of 3x3 matrix given by columns 1,3,4 is $2 \neq 0$ so these 3 vectors are independent.

v_2 is $-v_1$, so span is at most 3 dimensional. above matrix has nonzero 3x3 det, so span is at least 3 dimensional. Thus span <u>is</u> 3-dimensional.

Continued on page: 2

Problem: 1

What if I messed up the row reduction, and my vectors give a different subspace than V?

$v_1 = (1,-1,0,0)$
$v_3 = (1,-1,1,-1)$
$v_4 = (-1,1,1,1)$

could have just crossed out v_2, knowing rank is 3 and $v_2 = -v_1$.
Would have been safer, no danger of "moving" V during row reduction.

$w_1 = (1,-1,0,0)$
$w_2 = (0,0,1,-1)$
$w_3 = (0,0,1,1)$

$w_1 = v_1$	$v_1 = w_1$
$w_2 = v_3 - v_1$	$v_3 = w_2 + w_1$
$w_3 = v_4 + v_1$	$v_4 = w_3 - w_1$

So my subspace is V

//

Continued on page: X

[2] Let A be the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix}.$$

Compute the row space and column space of A.

A is 3×4 so rank is at most $\min(3,4) = 3$. Is it 3?

$$\begin{bmatrix} 2 & -1 & 0 & \cancel{-1} \\ -1 & 2 & -1 & \cancel{0} \\ 0 & -1 & 2 & \cancel{-1} \end{bmatrix}$$ look at 3×3 det of columns 1,2,3

$$= 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} = 2\cdot 3 + 1\cdot(-2) = 4 \neq 0$$

Thus rows 1,2,3 of A are a basis for rowspace:

$\boxed{\text{row space basis} = \{(2,-1,0,-1), (-1,2,-1,0), (0,-1,2,-1)\}}$

.. and cols 1,2,3 of A are a basis for columnspace

$\boxed{\text{col space basis} = \{(2,-1,0), (-1,2,-1), (0,-1,2)\}}$

but wait! This is a 3-dim subspace of $\mathbb{R}^3$, it must be $\mathbb{R}^3$.

$\boxed{\therefore \text{ col space is } \mathbb{R}^3}$

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checks: row reduce A

$$\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \end{bmatrix} \xrightarrow{1 \rightleftarrows 2} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & -1 & 0 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix} \xrightarrow{2 = 2 + 2\cdot 1} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 3 & -2 & -1 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$$\xrightarrow{2 \rightleftarrows 3} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 3 & -2 & -1 \end{bmatrix} \xrightarrow{3 = 3 + 3\cdot 2} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$

clearly independent so rank is indeed 3.

determinant of cols 1,2,3 still 4.

(caught earlier mistake while checking)

Page 3

Continued on page: ~~X~~

[3] Let V be the vector space of all polynomials $f(x)$ of degree ≤ 3. Let $W \subset V$ be the set of all polynomials f in V which satisfy $f'(1) = 0$. Show that W is a subspace of V. ① Find a basis for W. ② Extend this basis to a basis for V. ③

① Need to check $f, g \in W \Rightarrow f+g \in W,\ rf \in W$.

$f, g \in W \Rightarrow f'(1)=0,\ g'(1)=0 \Rightarrow (f+g)'(1) = f'(1) + g'(1) = 0+0=0$
$\Rightarrow f+g \in W.$

$f \in W \Rightarrow f'(1)=0 \Rightarrow (rf)'(1) = rf'(1) = r \cdot 0 = 0 \Rightarrow rf \in W$

So $\boxed{W \text{ is a subspace.}}$

② $V = \{ax^3 + bx^2 + cx + d\}$ write elements as (a,b,c,d)

$f(x) = ax^3 + bx^2 + cx + d$
$f'(x) = 3ax^2 + 2bx + c$
$f'(1) = 3a + 2b + c = 0$

$f'(1)=0$ is one equation on a 4-dimensional space, so W is a 3-dimensional subspace. Find 3 vectors in W, for basis.

$(\underbrace{0,0,0}_{3a+2b+c=0},1)$ $(\underbrace{2,-3,0}_{3a+2b+c=0},0)$ $(0,\underbrace{1,-2}_{3a+2b+c=0},0)$

So $\boxed{\text{basis for } W \text{ is } \{1,\ 2x^3-3x^2,\ x^2-2x\}}$

③ Any vector not lying in W will extend this 3-dim subspace to all of 4-dim V.

We know how to check: Any vector so $3a+2b+c \neq 0$ doesn't lie in W.

$(\underbrace{1,0,0}_{3a+2b+c=3\neq 0},0)$

$$\boxed{\text{So basis extending } W \text{ to } V \text{ is } \{\underbrace{1,\ 2x^3-3x^2,\ x^2-2x}_{\text{basis for } W},\ \underbrace{x^3}_{\text{extension to } V}\}}$$

Continued on page: 6

—over—

Problem: 3

checks:

$\{1,\ 2x^3-3x^2,\ x^2-2x\}$ basis for W

$$\begin{array}{ccc} 1 & 2x^3-3x^2 & x^2-2x \\ \Downarrow & \Downarrow & \Downarrow\ f'(x) \\ 0 & 6x^2-6x & 2x-2 \\ \Downarrow & \Downarrow & \Downarrow\ x=1 \\ 0 & 0 & 0 \end{array}$$

so these vectors are in W

$$\begin{vmatrix} & 1 \\ 2x^3-3x^2 & \\ x^2-2x & \end{vmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix}$$

vectors have distinct starting positions, so would be row reduced after sorting by start position,

so they are independent.

$f'(1)=0$ is one equation, so we expect a basis of 3 vectors. //

$\{1,\ 2x^3-3x^2,\ x^2-2x,\ x^3\}$ basis for V

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ \cancel{2} & 3 & 0 & 0 \\ 0 & * & -2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

can see row reduction of $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & -3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

will show rank 4.

(Would have been easier if I'd chosen x as last vector, then everyone gets different starting positions, clearly independent

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[4] Let $\underbrace{\mathbf{v}_1 = (2,1) \text{ and } \mathbf{v}_2 = (1,2)}_{V}$. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that

$$L(\mathbf{v}_1) = \mathbf{v}_1, \quad L(\mathbf{v}_2) = 2\mathbf{v}_2.$$

Find a matrix that represents L with respect to the usual basis $\underbrace{\mathbf{e}_1 = (1,0),\ \mathbf{e}_2 = (0,1)}_{E}$.

$$\underset{E\leftarrow E}{\overset{L}{\begin{bmatrix} & \\ & \end{bmatrix}}} = \underset{E\leftarrow V}{\overset{\text{indentity}}{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}} \underset{V\leftarrow V}{\overset{L}{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}} \underset{V\leftarrow E}{\overset{\text{indentity}}{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}} \Big/ 3$$

fill in cols using V

inverse of

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix} \Big/ 3 = \boxed{\begin{bmatrix} 2 & 2 \\ -2 & 7 \end{bmatrix} \Big/ 3 \quad \text{answer}}$$

check:

$$\begin{bmatrix} 2 & 2 \\ -2 & 7 \end{bmatrix} \Big/ 3 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 3 & 12 \end{bmatrix} \Big/ 3 = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ v_1 & v_2 \end{array} \qquad \qquad \begin{array}{cc} \uparrow & \uparrow \\ v_1 & 2v_2 \end{array}$$

works as advertised.

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[5] Let

$$\mathbf{v}_1 = (1,1,0), \quad \mathbf{v}_2 = (1,0,1), \quad \mathbf{v}_3 = (0,1,1). \qquad V$$

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$L(\mathbf{v}_1) = \mathbf{v}_3, \quad L(\mathbf{v}_2) = \mathbf{v}_1, \quad L(\mathbf{v}_3) = \mathbf{v}_2.$$

Find a matrix that represents L with respect to the usual basis

$$\mathbf{e}_1 = (1,0,0), \quad \mathbf{e}_2 = (0,1,0), \quad \mathbf{e}_3 = (0,0,1). \qquad E$$

$$\underset{E \leftarrow E}{\overset{L}{\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}}} = \underbrace{\underset{E \leftarrow V}{\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}}}_{\text{cols of } V} \underset{V \leftarrow V}{\overset{L}{\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}}} \underbrace{\underset{V \leftarrow E}{\begin{bmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{bmatrix}}}_{\text{inverse computed on other side}} /2$$

$$\cancel{= \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}/2 = \cdots}$$

copying mistake (something felt wrong)

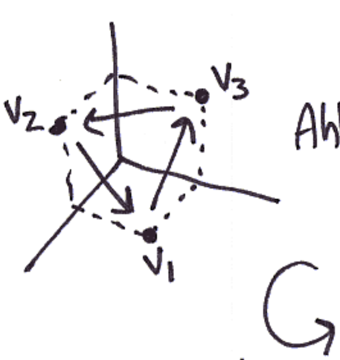
$$= \begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&-1&1\\-1&1&1\\1&1&-1\end{bmatrix}/2 = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

answer

(Note to myself: I did divide by 2 I know it looks like I forgot.)

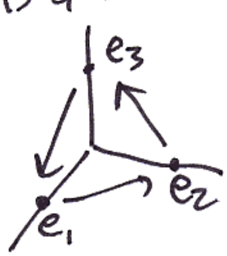
check:

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\underset{v_1\ v_2\ v_3}{\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}} = \underset{v_3\ v_1\ v_2}{\begin{bmatrix}0&1&1\\1&1&0\\1&0&1\end{bmatrix}} \quad ✓$$



1/3 turn

Ahh! L is a 3rd turn of first octant, so of course,



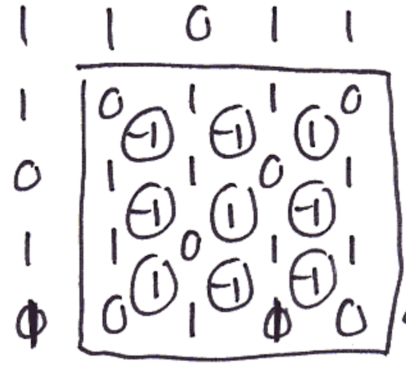
$$\underset{E \leftarrow E}{\overset{L}{\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}}}$$

Could have seen directly this is matrix //

Continued on page: 10

Problem: 5

$$\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}^{-1} \Rightarrow$$



caught this copying mistake because pattern didn't feel right

$$= \begin{bmatrix}-1&-1&1\\-1&1&-1\\1&-1&-1\end{bmatrix}/(-2) = \underbrace{\begin{bmatrix}1&1&-1\\1&-1&1\\-1&1&1\end{bmatrix}/2}_{\text{easier with fewer signs}}$$

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above is my new method for fast computation of 3x3 inverses by hand (see "inverse.pdf" on our course web page)

1. Transpose matrix
2. copy over first two cols, on right
3. copy over first two rows, on bottom. Find a way you like.
   (Steps 1, 2, 3 can be combined in various ways. I actually copy 1st row as 1st col, and "wrap around" till I write 5 numbers. I keep going, and copy first 2 cols over...)
4. Draw box around all but 1st row and col
5. Write in 2x2 dets in between all 2x2 blocks in box
   (don't correct any signs. This method has signs built in!)
6. Copy out these numbers as inverse, and correct denominator when checking answer. (This computes determinant.)

Continued on page: X