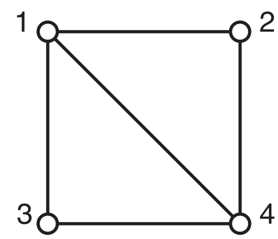
Practice Exam 1

[**1**] Solve the following system of equations:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 6 \end{bmatrix}$$

[**2**] Compute a matrix giving the number of walks of length 4 between pairs of vertices of the following graph:



[**3**] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

[**4**] Compute the determinant of the following 4×4 matrix:

$$\begin{bmatrix} \lambda & 1 & 0 & 0 \\ 1 & \lambda & 1 & 0 \\ 0 & 1 & \lambda & 1 \\ 0 & 0 & 1 & \lambda \end{bmatrix}$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

[**5**] Use Cramer's rule to give a formula for w in the solution to the following system of equations:

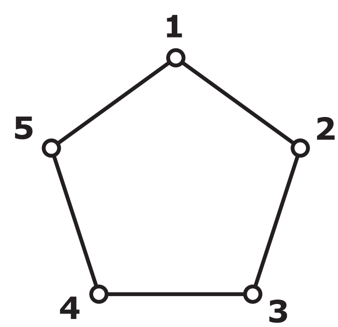
$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Exam 1

[1] Solve the following system of equations:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

[2] Compute matrices giving the number of walks of lengths 1, 2, and 3 between pairs of vertices of the following graph:



[3] Express the following matrix as a product of elementary matrices:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

[4] Compute the determinant of the following 4×4 matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 3 & 0 & 3 & 3 \\ 0 & 4 & 4 & 4 \end{bmatrix}$$

What can you say about the determinant of the $n \times n$ matrix with the same pattern?

[5] Use Cramer's rule to give a formula for the solution to the following system of equations:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \\ 2c \end{bmatrix}$$

Practice Exam 2

[**1**] Let P be the set of all polynomials $f(x)$, and let Q be the subset of P consisting of all polynomials $f(x)$ so $f(0) = f(1) = 0$. Show that Q is a subspace of P.

[**2**] Let A be the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 0 \end{bmatrix}.$$

Compute the row space and column space of A.

[**3**] The four vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

span a subspace V of $\mathbb{R}^3$, but are not a basis for V. Choose a subset of $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4\}$ which forms a basis for V. Extend this basis for V to a basis for $\mathbb{R}^3$.

[**4**] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which rotates one half turn around the axis given by the vector $(1, 1, 1)$. Find a matrix A representing L with respect to the standard basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Choose a new basis $\{\mathbf{v}_1,\ \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$ which makes L easier to describe, and find a matrix B representing L with respect to this new basis.

[**5**] Let $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ and $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, and let L be the linear transformation represented by the matrix A with respect to $\{\mathbf{e}_1,\ \mathbf{e}_2\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad A = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}.$$

Find the transition matrix S corresponding to the change of basis from $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$. Find a matrix B representing L with respect to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$.

Exam 2

[**1**] Let P be the set of all degree ≤ 4 polynomials in one variable x with real coefficients. Let Q be the subset of P consisting of all odd polynomials, i.e. all polynomials $f(x)$ so $f(-x) = -f(x)$. Show that Q is a subspace of P. Choose a basis for Q. Extend this basis for Q to a basis for P.

[**2**] Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}.$$

Compute the row space and column space of A.

[**3**] Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which reflects through the plane P defined by $x + y + z = 0$. In other words, if $\mathbf{u}$ is a vector lying in the plane P, and $\mathbf{v}$ is a vector perpendicular to the plane P, then $L(\mathbf{u} + \mathbf{v}) = \mathbf{u} - \mathbf{v}$. Choose a basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for $\mathbb{R}^3$, and find a matrix A representing L with respect to this basis.

[**4**] Let $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ and $\{\mathbf{v}_1,\ \mathbf{v}_2\}$ be ordered bases for $\mathbb{R}^2$, and let L be the linear transformation represented by the matrix A with respect to $\{\mathbf{e}_1,\ \mathbf{e}_2\}$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix}.$$

Find the transition matrix S corresponding to the change of basis from $\{\mathbf{e}_1,\ \mathbf{e}_2\}$ to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$. Find a matrix B representing L with respect to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$.

[**5**] Let $\{\mathbf{u}_1,\ \mathbf{u}_2\}$, $\{\mathbf{v}_1,\ \mathbf{v}_2\}$, and $\{\mathbf{w}_1,\ \mathbf{w}_2\}$ be ordered bases for $\mathbb{R}^2$. If

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

is the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ to $\{\mathbf{v}_1,\ \mathbf{v}_2\}$, and

$$B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

is the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1,\ \mathbf{u}_2\}$ to $\{\mathbf{w}_1,\ \mathbf{w}_2\}$, express $\mathbf{v}_1$ and $\mathbf{v}_2$ in terms of $\mathbf{w}_1$ and $\mathbf{w}_2$.

Additional Practice Problems for Final

[**1**] By least squares, find the equation of the form $y = ax + b$ which best fits the data $(x_1, y_1) = (0, 1)$, $(x_2, y_2) = (1, 1)$, $(x_3, y_3) = (2, -1)$.

[**2**] Find (s, t) so $\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$ is as close as possible to $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

[**3**] Find an orthogonal basis for the subspace $w + x + y + z = 0$ of $\mathbb{R}^4$.

[**4**] Let A be the matrix

$$A \; = \; \begin{bmatrix} -3 & -4 \\ -4 & 3 \end{bmatrix}.$$

Find a basis of eigenvectors and eigenvalues for A. Find the matrix exponential e^A.

[**5**] Find a matrix A in standard coordinates having eigenvectors $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (1, 2)$ with corresponding eigenvalues $\lambda_1 = 2$, $\lambda_2 = -1$.

[**6**] Let A be the matrix

$$A \; = \; \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Find an orthogonal basis in which A is diagonal.