ON A KUZNETSOV TRACE FORMULA WITHOUT CONTINUOUS SPECTRUM

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Abstract. This paper constructs a new Kuznetsov trace formula for $\mathrm{SL}(2,\mathbb{Z})$ Maass forms that is absent of a contribution from the continuous spectrum. As an application, we demonstrate non-trivial cancellation of spectral oscillatory sums convolved with GL(2) Fourier coefficients.

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1. Introduction

The Kuznetsov trace formula for $\mathrm{SL}(2,\mathbb{Z})$ Maass forms, first conceived in [Kuz89], is an equality of two expressions: one consisting of the Fourier coefficients of Maass forms, and the other consisting of sums of Kloosterman sums mixed with some smooth weights. Recently, the works [Gol20, Theorem 9.1] [GSW25] introduced a new version of the Kuznetsov trace formula for $\mathrm{SL}(2,\mathbb{Z})$. We will elucidate its form in the display below, and then introduce its notation that will be used in the sequel:

$$\sqrt{mn}\sum_{r_j}\lambda_j(m)\lambda_j(n)\cdot\frac{|p^{\#}(ir_j)|^2}{\langle\phi_j,\phi_j\rangle}+\{\text{contribution from Eisenstein series}\} \tag{1.0.1}$$

$$=\mathbb{1}_{m=n}\int_0^\infty p(my)q(ny)\frac{dy}{y^2}+\sum_{c=1}^\infty S_1(m,n;c)\int_0^\infty\int_{-\infty}^\infty p\left(\frac{m}{c^2y(x^2+1)}\right)p(ny)e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\frac{dxdy}{y} \tag{1.0.2}$$

where

- We fix $m,n\in\mathbb{Z}$ and $p(y)$ is a real-valued function such that

$$p(y)\ll\begin{cases}y^{1+\varepsilon} & 0\le y\le 1\\ y^{-C} & y>1\end{cases},$$

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for some $\varepsilon > 0$ and $C > 1$.

- The ϕ_j are an orthonormal basis of eigenfunctions for the Laplacian $\Delta = -y^2\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)$ on the upper half-plane $\{z = x + iy : y > 0\}$ such that its first Fourier coefficient is 1.
- The ϕ_j have eigenvalues $\frac{1}{4} + r_j^2$, with $r_j \in \mathbb{R}$ (this is known since Δ is self-adjoint [Gol06a, Propostion 3.3.2]), ordered so that $\{r_j\}$ is increasing and $r_1 > 0$.
- The ϕ_j have Fourier coefficients $\lambda_j(\cdot)$.
- We will set $S_\kappa(m, n; c) := \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} e^{\frac{a\kappa^{-1}n + \overline{a}m}{c}}$, where $a\overline{a} \equiv 1 \pmod{c}$ and $\kappa \in \mathbb{Q}$. When $\kappa = 1$, this is the classical $\mathrm{SL}(2, \mathbb{Z})$ Kloosterman sum.
- The function $p^\#(ir)$ is the *Bessel transform* of $p(y)$, where

$$p^\#(ir) := \int_0^\infty p(y) K_{ir}(2\pi y) \frac{dy}{y^{3/2}}$$

and $K_{ir}(y) = \frac{1}{2}\int_0^\infty e^{-\frac{u}{2}\left(u + \frac{1}{u}\right)} u^{ir} \frac{du}{u}$ is the K-Bessel function.

The Bessel transform has an inverse transform that will be indispensible for this paper [GKS11]:

$$p(y) = \frac{1}{\pi}\int_{-\infty}^\infty p^\#(ir)\, \sqrt{y}\, K_{ir}(2\pi y) \frac{dr}{\Gamma(ir)\Gamma(-ir)}. \tag{1.0.3}$$

We note for comparison that (1.0.1) uses this inverse transform on its right-hand side. In contrast, the analogous term on the right-hand side of the Kuznetsov trace formula [Kuz89] integrates against the distribution $r \tanh(r) dr$. In [Gol20], this buys an integral representation of $p(y)$ in terms of Gamma functions, and choosing $p^\#(ir)$ to be a simple cutoff function for the spectral window $[0, T]$ allows one to obtain an asymptotic formula with a power-savings error term for

$$\sum_{r_j} \lambda_j(m)\lambda_j(n) \cdot \frac{|p^\#(ir_j)|^2}{\langle \phi_j, \phi_j \rangle},$$

where m and n are fixed.

This paper demonstrates a variant of Goldfeld's formulation that is capable of achieving an asymptotic with power-savings error term for triple products of Fourier coefficients, where one of the coefficients is shifted by a phase.

By a careful analysis of this formula, we are able to obtain asymptotics for sums of k-products of Fourier coefficients in the spectral windows $I_\delta(T) := [T - T^\delta, T + T^\delta]$ where $0 < \delta < 1$. These will correspond to the cutoffs

$$p^\#_{T,R,\delta}(\alpha) := e^{\frac{(\alpha - iT)^2}{T^{2\delta}}} \Gamma\left(\frac{2 + R + 2\alpha}{4}\right)\Gamma\left(\frac{2 + R - 2\alpha}{4}\right). \tag{1.0.4}$$

Precisely, we prove that

Theorem 1.1. *Let $k \geq 2$ be an integer and $m, n, \eta \in \mathbb{Z}_{>0}$, where η is prime. Fix $0 < \delta < 1$ such that $\delta \neq 1/3$. Then, for any $\varepsilon > 0$, we have that*

$$\sum_{r_j} \lambda_j(m)\lambda_j(n)(\lambda_j(\eta) - \cos(r_j \log \eta)) \cdot \frac{p^\#_{T,R,\delta}(ir_j)}{\langle \phi_j, \phi_j \rangle} \asymp T^{R+1+\delta} \mathbb{1}_{\exists d | (m,\eta) : m\eta d^{-2} = n} + O(T^{R+2\delta+\varepsilon})\mathbb{1}_{m=n} + O(T^{\frac{R+1}{2}+2\delta} M),$$

where the $\mathbb{1}_{m=n}O(T^{R+2\delta+\varepsilon})$ *may be replaced by* $\mathbb{1}_{m=n}O(T^{R+1-\delta}\log T)$ *when* $\delta < 1/3$ *and*
$M = \min\{(mn\eta)^{R/2+1/2+\varepsilon}, (mn)^{R/2+1/2+\varepsilon}+\eta^{R/2+1/2+\varepsilon}, (m\eta)^{R/2+1/2+\varepsilon}+n^{R/2+1/2+\varepsilon}, (n\eta)^{R/2+1/2+\varepsilon}+m^{R/2+1/2+\varepsilon}\}$.

The novelty of our method is that we obtain non-trivial cancellation in all sums below for any $0 < \delta < 1$ where $\delta \neq 1/3$. Moreover, the cancellation is near square-root when $\delta \to 1/3$ from the left and as $\delta \to 1$.

Corollary 1.1.1. *Let variables be as above. Then, for any* $\varepsilon > 0$*, we obtain*

$$\sum_{r_j}{}^{\#} \cos(r_j \log \eta) \cdot \frac{1}{L(1, \mathrm{Ad}\phi_j)} \ll_\varepsilon T^{1-\delta} + \mathbb{1}_{\delta>1/3}T^{2\delta+\varepsilon}.$$

Proof. In Theorem 1.1, let $m = n = 1$. Because $\delta < 1/3$, we have

$$\sum_{r_j}(\lambda_j(\eta) - \cos(r_j \log \eta)) \cdot \frac{p^{\#}_{T,R,\delta}(ir_j)}{\langle \phi_j, \phi_j\rangle} \ll T^{R+1-\delta} + O(T^{R+2\delta-3/2}M). \tag{1.1.1}$$

Now, we may estimate $\sum_{r_j} \lambda_j(\eta) \cdot \frac{p^{\#}_{T,R,\delta}(ir_j)}{\langle \phi_j, \phi_j\rangle}$ by a modification of the Kuznetsov trace formula of [Gol20, Theorem 9.16]. The Eisenstein contribution there will be multiplied in our case by $T^{\delta-1}$, since the Gaussian factor of $p^{\#}_{T,R,\delta}(ir)$ has length T^δ and the error term remains the same as our M, but there we take $\eta = 1$ in M. Thus, we have that

$$\sum_{r_j} \lambda_j(\eta) \cdot \frac{p^{\#}_{T,R,\delta}(ir_j)}{\langle \phi_j, \phi_j\rangle} \ll T^{R+\delta} + O(T^{R+2\delta-3/2}M),$$

which by (1.1.1) implies

$$\sum_{r_j} \cos(r_j \log \eta) \cdot \frac{p^{\#}_{T,R,\delta}(ir_j)}{\langle \phi_j, \phi_j\rangle} \ll T^{R+1-\delta} + O(T^{R+2\delta-3/2}M).$$

We have that $p^{\#}_{T,R,\delta}(ir_j)$ is asymptotic to $T^R e^{-\pi r_j/2}$ in the support of the Gaussian term of $p^{\#}_{T,R,\delta}(ir_j)$. Divide this away from the above inequality and using the fact that $1 - \delta > \delta$, we obtain that

$$\sum_{r_j}{}^{\#} \cos(r_j \log \eta) \cdot \frac{1}{L(1, \mathrm{Ad}\phi_j)} \ll T^{1-\delta} + O(T^{2\delta-3/2}M),$$

where $\sum_{r_j}{}^{\#}$ denotes the sum with a smooth bump function with cutoff at $T - T^\delta$ and $T + T^\delta$. □

Remark 1.1.1. *We have not tried to optimize the size that* η *can be. Note that, by the remark following Corollary 2.9.1, we may remove the condition that* η *is prime in exchange for* $\cos(r_j \log \eta)$ *being replaced with* $\frac{1}{2}\sigma_{-2ir_j}(\eta)$*.*

The philosophy here in general is that, if m, n, η do not satisfy Hecke relations, then we obtain near square-root cancellation as $\delta \to 1/3$, and we find trivial bounds otherwise. Below, we give a few more results illustrating this. The proofs are exactly the same as the previous corollary.

Corollary 1.1.2. *Let variables be as above and* $\delta < 1/3$*. Then, for any* $\varepsilon > 0$ *and on the condition that* $(m, \eta) = 1$*, we have*

$$\sum_{r_j}{}^{\#} \lambda_j(m) \cos(r_j \log \eta) \cdot \frac{1}{L(1, \mathrm{Ad}\phi_j)} \ll_\varepsilon T^{1-\delta} + \mathbb{1}_{\delta>1/3}T^{2\delta+\varepsilon}.$$

Corollary 1.1.3. *Let variables be as above and $\delta < 1/3$. Choose $m = n$ square-free. Then, for any $\varepsilon > 0$, we have that*

$$\sum_{r_j}{}^{\#}|\lambda_j(m)|^2 \cos(r_j \log \eta) \cdot \frac{1}{L(1, \mathrm{Ad}\phi_j)} \ll T^{1-\delta} + \mathbb{1}_{\delta>1/3} T^{2\delta+\varepsilon}.$$

All the previous results could hold, pending analysis on the geometric side. when you replace $\cos(r_j \log \eta)$ by $\sin(r_j \log \eta)$.

Thus, the bounds on η represent progress to a problem posed in [ST10, Eq. (72)], which desires the size of η to be T^3. It is likely that a better understanding of the inverse Bessel transform will give further progress towards achieving this range that will occur in forthcoming work [Zhab]. We also want to highlight future work generalizing Appendix A, which will allow us to make statements such as the following

Theorem 1.2 ([Zhaa])**.** *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a periodic L^1-function with Fourier coefficients of sufficient decay. If $m = n$ square-free with $M \ll T^{R/4+1-3\delta}$, we have, for any $\varepsilon > 0$, that*

$$\sum_{r_j}{}^{\#}\lambda_j(m)\lambda_j(n)\frac{f(r_j \log \eta)}{L(1, \mathrm{Ad}\phi_j)} \ll T^{1-\delta} + \mathbb{1}_{\delta>1/3} T^{2\delta+\varepsilon}.$$

Finally, we remark that, for the corollaries to our main Theorem 1.1, it would have sufficed to only consider the effect of the operator $U_{\log \eta}$ defined in §2, instead of the difference $T_\eta - U_{\log \eta}$. We opt for the latter to show that we can obtain a Kuznetsov trace formula without continuous spectrum. We hope that, should we be able to find an automorphic interpretation of the difference $\lambda_j(\eta) - \cos(r_j \log \eta)$, that this approach will become very useful.

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2. Prelude to a New Kuznetsov Trace Formula

This section will introduce preliminary notions for the triple Kuznetsov trace formula. We start by defining non-holomorphic Poincaré series.

Definition 2.1. *Let $p : \mathbb{R} \to \mathbb{C}$ be a smooth function satisfying*

$$p(y) \ll \begin{cases} y^{1+\varepsilon} & 0 < y \leq 1, \\ y^{-C} & y > 1, \end{cases}$$

for some $\varepsilon > 0$ and fixed constant $C > 1$. Define the non-holomorphic Poincaré series to be the function given by

$$\mathcal{P}_m(z, p) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} p(m \cdot \mathrm{Im}(\gamma z))\, e^{2\pi i m \cdot \mathrm{Re}(\gamma z)}.$$

Let us now define our two main operators.

Definition 2.2. *Let T_η be the Hecke operator at the prime η. Moreover, the operator $\frac{1}{2}U_{\log\eta}(f)$ is the solution at time $t = \log\eta$ to the wave equation $u_{tt} = \Delta u + \frac{u}{4}$, where Δ is the hyperbolic Laplacian, with initial data $u_0 = f$ and $\frac{d}{dt}\Big|_{t=0} u_t = 0$.*

We record two properties of the $U_{\log\eta}$ operator that will be useful for later.

Lemma 2.3. *The operator $U_{\log\eta}$ is linear. Moreover, for any integer $\eta = \prod_i p_i^{m_i}$, the operator $U_{\log\eta} := \circ_i U_{\log p_i}^{m_i}$ is well-defined on the continuous spectrum.*

Proof. By [LV07, §1], the operator $U_{\log\eta}$ may be understood as a convolution operator via the spherical transform, which is linear. We can also see this by the explicit description above since u_t is determined by the prescribed initial data and the fact that $\frac{d}{dt}\Big|_{t=0} u_t = 0$. For the second claim, by *loc cit.*, $E\left(z, \frac{1}{2} + ir\right)$ is an eigenfunction of $U_{\log\eta}$ with eigenvalue $\frac{1}{2}(\eta^{ir} + \eta^{-ir})$. □

Lemma 2.4. *Both T_η and $U_{\log\eta}$ are self-adjoint and bounded, and hence $T_\eta - U_{\log\eta}$ is self-adjoint. Moreover, the difference $T_\eta - U_{\log\eta}$ annihilates the Eisenstein series.*

Proof. The claim for T_η is well-known and is known for $U_{\log\eta}$ by [LV07, §1], and hence the composition $U_{\log\eta}$. The deduction that $T_\eta - U_{\log\eta}$ is standard. The eigenvalue of $E\left(z, \frac{1}{2} + ir\right)$ under $U_{\log\eta}$ obtained from [LV07, §1] matches the ηth Fourier coefficient of $E\left(z, \frac{1}{2} + ir\right)$, written out in [Gol06b, Theorem 3.1.8]. □

Let the cuspidal part be spanned by Hecke eigenforms ϕ_j with eigenvalue $\frac{1}{4}+r_j^2$ for the Laplacian. Then, $U_{\log\eta}$ sends ϕ_j in the discrete spectrum to $(\lambda_\eta(\phi) - (\eta^{ir_j} + \eta^{-ir_j})) \cdot \phi_j$. We make the following

Definition 2.5. *Let $R_j(\eta) := \lambda_\eta(\phi_j) - \frac{\eta}{2}\sigma_{-2ir_j}(\eta)$ be the eigenvalue of ϕ_j for the operator $T_\eta - U_{\log\eta}$, where σ_{-2ir_j} the $(-2ir_j)^{\text{th}}$ power divisor sum and is $\frac{\eta}{2}\sigma_{-2ir_j}(\eta) = \cos(r_j \log\eta)$ for η prime.*

With the preliminaries in place, we will obtain the Hecke–Kuznetsov trace formula by expanding the expression

$$\left\langle \left(T_\eta - U_{\log\eta}\right)\mathcal{P}_m(z,p),\ \mathcal{Q}_n(z,q)\right\rangle \tag{2.5.1}$$

spectrally and by unfolding with the Fourier expansion of $\mathcal{Q}_n(z,q)$. We start with the former.

2.1. **Spectral Expansion.** Let us now expand $\left\langle (T_\eta - U_{\log\eta})\mathcal{P}_m(z,p),\ \mathcal{Q}_n(z,q)\right\rangle$ by spectrally expanding $\mathcal{Q}(z,q)$ with respect to $\Delta = -y^2\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)$. This is possible since $\mathcal{Q}_n(z,q) \in L^2(\Gamma\backslash\mathbb{H})$:

$$\begin{aligned}|\mathcal{Q}_n(z,q)| &\ll\ q(y)\ +\sum_{I_2\neq\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^{1+\varepsilon}\\ &\ll 1.\end{aligned}$$

We then obtain that

$$\begin{aligned}\left\langle \left(T_\eta - U_{\log\eta}\right)\mathcal{P}_m(z,p),\ \mathcal{Q}_n(z,q)\right\rangle &= \sum_{j=0}^{\infty} \frac{\left\langle \left(T_\eta - U_{\log\eta}\right)\mathcal{P}_m(z,p),\ \phi_j\right\rangle \cdot \left\langle \phi_j,\ \mathcal{Q}_n(z,q)\right\rangle}{\langle\phi_j,\phi_j\rangle}\\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty} \left\langle \left(T_\eta - U_{\log\eta}\right)\mathcal{P}_m(z,p),\ E(z,1/2+ir)\right\rangle \cdot \left\langle E(z,1/2+ir),\ \mathcal{Q}_n(z,q)\right\rangle dr.\end{aligned}$$

Since $(T_\eta - U_{\log \eta})$ is self-adjoint, the continuous contribution disappears, and we are left with

$$\sum_{j=0}^{\infty} \overline{R_j(\eta)} \cdot \frac{\langle \mathcal{P}_m(z,p), \phi_j \rangle \cdot \langle \phi_j, \mathcal{Q}_n(z,q) \rangle}{\langle \phi_j, \phi_j \rangle}.$$

Each ϕ_j has a Fourier expansion of the form

$$\phi_j(z) = \sum_{k \neq 0} \lambda_j(k) \sqrt{y} K_{ir_j}(2\pi |k| y) e^{2\pi i k x}$$

with the K-Bessel function. We then record the following well-known result, which follows from unfolding the Poincaré series.

Lemma 2.6. *One has that*

$$\left\langle \mathcal{P}_m(z,p),\ \phi_j \right\rangle = \begin{cases} 0 & j = 0, \\ m^{1/2} \lambda_j(m) \int_0^\infty p(y) K_{ir_j}(2\pi y) \frac{dy}{y^{3/2}} & j > 0. \end{cases}$$

Proof. See [Gol20, Lemma 7.2]. □

Incorporating this into the expansion, we obtain

Proposition 2.7. *The spectral side of the η-twisted Kuznetsov trace formula is given by*

$$\sqrt{mn} \sum_{j=1}^{\infty} R_j(\eta) \lambda_j(m) \lambda_j(n) \cdot \frac{p^\#(ir_j) \overline{q^\#(ir_j)}}{\langle \phi_j, \phi_j \rangle}.$$

2.2. **Geometric Expansion.** Let us now write what happens for the geometric expansion. We have the following structure of the double coset space $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$.

Lemma 2.8. *We have the following disjoint union:*

$$\Gamma = \Gamma_\infty \sqcup \left(\bigsqcup_{c>0} \bigsqcup_{d \in \mathbb{Z}/c\mathbb{Z}} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \right)$$

Proof. Omitted. See [Gol20, Lemma 8.1]. □

The geometric expansion hinges upon this decomposition, as it allows us to rewrite the Poincaré series $\mathcal{P}_m(z,p)$ as

$$\mathcal{P}_m(z,p) = p(my) e^{2\pi i m x} + \sum_{c=1}^{\infty} \sum_{d \in \mathbb{Z}}{}' p\left(\frac{my}{|cz+d|^2} \right) e^{2\pi i m \operatorname{Re}\left(\frac{az+b}{cz+d}\right)},$$

where $\sum'$ means to sum over integers coprime to c. Because of the double coset decomposition, we can choose a and b so long as $ad - bc = 1$.

We now need to evaluate how $(T_\eta - U_{\log \eta})$ affects the geometric side. We will devote one subsection to the effect of T_η and another subsection to the effect of $U_{\log \eta}$. We follow the method of [Gol20, §8].

2.2.1. *Effect of T_η.* For this section only, let η be any positive integer. We will evaluate the Fourier coefficients of $T_\eta \mathcal{P}_m$ and then unravel Q_n. It will be useful to define

$$S_\eta(m,n;c,\{f_i\}_{i\in(\mathbb{Z}/c\mathbb{Z})^\times}) := \sum_{\substack{0\le d<c\\ ad\equiv 1 \pmod c}} e^{2\pi i\frac{a\frac{\eta}{\eta}+dm}{c}} f_a$$

to be a modified Kloosterman sum. Also let $\mathbb{1}_{\mathcal{S}}$ denote the indicator function which is either 0 or 1 according as a statement $\mathcal{S}$ holds or not.

Proposition 2.9. *Fix $\eta, m, n \in \mathbb{Z}$ and let $f_{c,k,\delta}$ be the piecewise function defined in* (2.9.2)*. Then we have*

$$\sum_{\alpha\delta=\eta}\sum_{\overline{a}=0}^{\delta-1}\int_0^1 \mathcal{P}_m\left(\frac{\alpha z+\overline{a}}{\delta},p\right)e^{-2\pi i n x}dx = \sum_{\delta|\eta}\delta\cdot\mathbb{1}_{\frac{m\eta}{\delta^2}=n}\,\mathbb{1}_{\delta|m}\cdot p\left(\frac{m\eta y}{\delta^2}\right)$$
$$+\sum_{\delta|\eta}\frac{\delta}{\eta}\sum_{c=1}^{\infty}\int_{\mathbb{R}} p\left(\frac{m\eta y}{c^2\left(x^2+\left(\frac{\eta y}{\delta}\right)^2\right)}\right)e^{-2\pi i x\left(\frac{\delta^3 m y}{c^2(\delta^2x^2+\eta^2y^2)}\right)}S_{\frac{\eta}{\delta^2}}(m,n;c,f_{c,k,\delta}(x))\,dx$$

Proof. Follows by computation. Begin by simply expanding:

$$\sum_{\alpha\delta=\eta}\sum_{\overline{a}=0}^{\delta-1}\int_0^1 \mathcal{P}_m\left(\frac{\alpha z+\overline{a}}{\delta},p\right)e^{-2\pi i n x}dx = \sum_{\delta|\eta}\int_0^1 p\left(\frac{m\eta y}{\delta^2}\right)e^{2\pi i\left(\frac{m\eta}{\delta^2}-n\right)x}dx\cdot\left(\sum_{\overline{a}=0}^{\delta-1}e^{2\pi i m\overline{a}/\delta}\right)$$
$$+\sum_{\delta|\eta}\int_0^1\left[\sum_{\overline{a}=0}^{\delta-1}\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z},(c,d)=1} p\left(\frac{m\eta y}{\left|c\left(\frac{\eta}{\delta}z+\bar{a}\right)+\delta d\right|^2}\right)e^{2\pi i m\cdot\mathrm{Re}\left(\frac{a\left(\frac{(\eta/\delta)z+\overline{a}}{\delta}\right)+b}{c\left(\frac{(\eta/\delta)z+\overline{a}}{\delta}\right)+d}\right)}\right]e^{-2\pi i n x}dx. \tag{2.9.1}$$

Now, make note of the identity

$$\frac{az+b}{cz+d}=\frac{a}{c}-\frac{1}{c(cz+d)},\qquad \mathrm{Re}\left(\frac{az+b}{cz+d}\right)=\frac{a}{c}-\frac{cx+d}{c|cz+d|^2}$$

and let $d = rc + k$ (where $1 \le k \le c$). It follows that the sum over $\delta|\eta$ of the last integral in (2.9.1) is given by

$$\sum_{\delta|\eta}\sum_{\overline{a}=0}^{\delta-1}\sum_{c=1}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{(k,c)=1}\int_0^1 p\left(\frac{m\eta y}{\left|c\left(\frac{\eta}{\delta}z+\bar{a}\right)+\delta rc+\delta k\right|^2}\right)e^{2\pi i m\cdot\mathrm{Re}\left[\frac{a}{c}-\frac{1}{c}\left(c\left(\frac{(\eta/\delta)z+\overline{a})}{\delta}\right)+rc+k\right)^{-1}\right]}e^{-2\pi i n x}\,dx$$
$$=\sum_{\delta|\eta}\sum_{\overline{a}=0}^{\delta-1}\sum_{c=1}^{\infty}\sum_{r=-\infty}^{\infty}\sum_{(k,c)=1}\int_0^1 p\left(\frac{m\eta y}{\left|c\left(\frac{\eta}{\delta}z+\bar{a}\right)+\delta rc+\delta k\right|^2}\right)e^{2\pi i m\cdot\mathrm{Re}\left[\frac{a}{c}-\frac{\delta}{c}\left(c\left(\frac{\eta}{\delta}z+\bar{a}\right)+\delta rc+\delta k\right)^{-1}\right]}e^{-2\pi i n x}\,dx$$

Next, make the change of variable $x \mapsto \frac{\delta}{\eta}\left(x - \overline{a} - \delta\left(r + \frac{k}{c}\right)\right)$ and the above integral transforms to

$$\sum_{\delta|\eta} \frac{\delta}{\eta} \sum_{\overline{a}=0}^{\delta-1} \sum_{c=1}^{\infty} \sum_{r=-\infty}^{\infty} \sum_{(k,c)=1} \int_{-\frac{\overline{a}\delta}{\eta} - \frac{\delta^2}{\eta}\left(r+\frac{k}{c}\right)}^{-\frac{\overline{a}\delta}{\eta} - \frac{\delta^2}{\eta}\left(r+\frac{k}{c}\right)+\frac{\delta}{\eta}} p\left(\frac{m\eta y}{c^2\left(x^2 + \left(\frac{\eta y}{\delta}\right)\right)^2}\right) e^{2\pi i m\left(\frac{a}{c} - \frac{\delta^3 x}{c(\delta^2 x^2 + \eta^2 y^2)}\right)} e^{-2\pi i \frac{n\delta}{\eta}\left(x - \overline{a} - \delta\left(r+\frac{k}{c}\right)\right)} dx$$

$$= \sum_{\delta|\eta} \frac{\delta}{\eta} \sum_{c=1}^{\infty} \sum_{(k,c)=1} S_{\frac{\eta}{\delta^2}}(m,n;c) \sum_{\overline{a}=0}^{\delta-1} \sum_{r=-\infty}^{\infty} \int_{-\frac{\overline{a}\delta}{\eta} - \frac{\delta^2}{\eta}\left(r+\frac{k}{c}\right)}^{-\frac{\overline{a}\delta}{\eta} - \frac{\delta^2}{\eta}\left(r+\frac{k}{c}\right)+\frac{\delta}{\eta}} p\left(\frac{m\eta y}{c^2\left(x^2 + \left(\frac{\eta y}{\delta}\right)\right)^2}\right) e^{-2\pi i m\left(\frac{\delta^3 x}{c(\delta^2 x^2 + \eta^2 y^2)}\right)} e^{-2\pi i \frac{n\delta}{\eta}(x - \overline{a} - \delta r)} dx.$$

Observe that as $\overline{a}$ and r vary (with c, k, δ, η fixed), the union of all intervals we integrate over is exactly $\mathbb{R}$. Each such integral has length $\frac{\delta}{\eta}$. The only function in the integrand that depends on $\bar{a}$ or r is the exponential function $e^{-2\pi i \frac{n\delta}{\eta}(x - \overline{a} - \delta r)}$ which is periodic with period $\frac{\eta}{n\delta}$. However the value of this function depends on the interval of integration, so we will replace it with a piecewise function $f_{c,k,\delta}$ defined as follows. For x in the interval $\left[-\frac{\overline{a}\delta}{\eta} - \frac{\delta^2}{\eta}\left(r + \frac{k}{c}\right),\ -\frac{\overline{a}\delta}{\eta} - \frac{\delta^2}{\eta}\left(r + \frac{k}{c}\right) + \frac{\delta}{\eta}\right]$ (which depends only on $-\bar{a} - \delta r$) the piecewise function is defined by

$$f_{c,k,\delta}(x) := e^{-2\pi i \frac{n\delta}{\eta}(x - \overline{a} - \delta r)}. \tag{2.9.2}$$

It follows that

$$\sum_{\alpha\delta=\eta} \sum_{\overline{a}=0}^{\delta-1} \int_0^1 \mathcal{P}_m\left(\frac{\alpha z + \overline{a}}{\delta}, p\right) e^{-2\pi i n x} dx = \sum_{\delta|\eta} \int_0^1 p\left(\frac{m\eta y}{\delta^2}\right) e^{2\pi i\left(\frac{m\eta}{\delta^2} - n\right)x} dx \cdot \left(\sum_{\overline{a}=0}^{\delta-1} e^{2\pi i m\overline{a}/\delta}\right)$$

$$+ \sum_{\delta|\eta} \frac{\delta}{\eta} \sum_{c=1}^{\infty} \sum_{(k,c)=1} \int_{-\infty}^{\infty} p\left(\frac{m\eta y}{c^2\left(x^2 + \left(\frac{\eta y}{\delta}\right)\right)^2}\right) e^{-2\pi i m\left(\frac{\delta^3 x}{c(\delta^2 x^2 + \eta^2 y^2)}\right)} S_{\frac{\eta}{\delta^2}}(m,n;c, f_{c,k,\delta}(x))\, dx. \tag{2.9.3}$$

To complete the proof note that $\sum_{\overline{a}=0}^{\delta-1} e^{2\pi i m\overline{a}/\delta}$ vanishes unless $\delta|m$ in which case it is equal to δ and that the integral $\int_0^1 e^{2\pi i\left(\frac{m\eta}{\delta^2} - n\right)x} dx$ vanishes unless $\frac{m\eta}{\delta^2} = n$. □

Corollary 2.9.1. *Now assume η to be prime. The Hecke contribution to the geometric side of the Hecke–Kuznetsov trace formula is given by*

$$\left\langle T_\eta \mathcal{P}_m(z,p),\ \mathcal{Q}_n(z,q)\right\rangle = \sum_{\delta|\eta} \delta \cdot \mathbb{1}_{\frac{m\eta}{\delta^2}=n}\, \mathbb{1}_{\delta|m} \int_0^\infty p\left(\frac{m\eta y}{\delta^2}\right) \overline{q(ny)}\, \frac{dy}{y^2}$$

$$+ \sum_{c=1}^{\infty} \int_0^\infty \int_{\mathbb{R}} p\left(\frac{m\eta y}{c^2\left(x^2+y^2\right)}\right) e^{-2\pi i x\left(\frac{\eta}{c^2(x^2+y^2)}\right)} S_{\eta^{-1}}(m,n;c,1)\, \overline{q(ny)}\, \frac{dxdy}{y}$$

$$+ \eta^{-1} \cdot \sum_{c=1}^{\infty} \int_0^\infty \int_{\mathbb{R}} p\left(\frac{m\eta y}{c^2\left(x^2+\eta^2 y^2\right)}\right) e^{-2\pi i x\left(\frac{1}{c^2(x^2+\eta^2 y^2)}\right)} S_\eta(m,n;c, f_{c,k,1}(x))\, \overline{q(ny)}\, \frac{dxdy}{y}.$$

Remark 2.9.1. *Of course, one can remove the restriction that η is prime, and hence the hypotheses of Theorem 1.1 and its corollaries may be relaxed to allow η to be composite. We only state the formula for prime η for simplicity.*

2.3. **Effect of $U_{\log\eta}$.** We aim do the same thing as the previous subsection, replacing T_η by $U_{\log\eta}$ and following the calculations. Preparatory results can be found in Appendix A.

Proposition 2.10. *We have*

$$\int_0^1 U_{\log\eta}\Big(\mathcal{P}_m(z,p)\Big)e^{-2\pi inx}\,dx$$
$$= \delta_{m,n}\tilde{p}(my) \;+\; y\sum_{c=1}^{\infty} S_1(m,n;c,1)\int_{\mathbb{R}} U_{\log\eta}\left(p\left(\frac{m}{c^2y(x^2+1)}\right)e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\right)dx,$$

where $\tilde{p}$ is as in Lemma A.2.

Proof. Begin once again by expanding, justified by using Lemma A.3:

$$\begin{aligned}\int_0^1 U_{\log\eta}\Big(\mathcal{P}_m(z,p)\Big)\,e^{-2\pi inx}\,dx &= \int_0^1 U_{\log\eta}\Big(p(my)e^{2\pi imx}\Big)\,e^{-2\pi inx}\,dx\\ &\quad+\int_0^1\left[\sum_{c=1}^{\infty}\sum_{d\in\mathbb{Z},(c,d)=1} U_{\log\eta}\left(p\left(\frac{my}{|cz+d|^2}\right)e^{2\pi im\cdot\Re\left(\frac{az+b}{cz+d}\right)}\right)\right]e^{-2\pi inx}\,dx\\ &= U(m,n)+\mathcal{U}_\eta(z).\end{aligned}$$

From here on, follow the strategy of the proof of Proposition 2.9. We find that, by Lemma A.2,

$$U(m,n) = \delta_{mn}\tilde{p}(my)$$

and

$$\mathcal{U}_\eta(z) = y\cdot\sum_{c=1}^{\infty} S_1(m,n;c,1)\int_{\mathbb{R}} U_{\log\eta}\left(p\left(\frac{m}{c^2y(x^2+1)}\right)e^{-2\pi ix\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\right)dx,$$

where, to derive the second equation, we use the linearity of $U_{\log\eta}$. □

2.4. **Bringing it Altogether.** We hence obtain the following η-twisted Kuznetsov trace formula:

Theorem 2.11. *Let $p,q:\mathbb{R}\to\mathbb{C}$ be smooth functions satisfying*

$$|p(y)|,|q(y)| \ll \begin{cases} y^{1+\varepsilon} & 0<y\le 1\\ y^{-C} & y>1,\end{cases}$$

and

$$|p'(y)|,|q'(y)| \ll \begin{cases} y^{2+\varepsilon} & 0<y\le 1\\ y^{-C-1} & y>1,\end{cases}$$

for some $\varepsilon>0$ and some $C>1$. Define the Bessel transforms

$$p^{\#}(ir) := \int_0^\infty p(y)K_{ir}(2\pi y)\,\frac{dy}{y^{3/2}},\qquad q^{\#}(ir) := \int_0^\infty q(y)K_{ir}(2\pi y)\,\frac{dy}{y^{3/2}}.$$

Let

$$\phi_j(z) = \sqrt{y}\sum_{k\neq 0} A_j(k)\, K_{ir_j}(2\pi|k|y)\, e^{2\pi i k x}, \qquad (j = 1, 2, \dots)$$

denote an orthonormal basis of Maass cusp forms, where

$$\Delta\phi_j = \left(\tfrac{1}{4} + r_j^2\right)\phi_j.$$

Then, the Hecke–Kuznetsov trace formula is the following identity:

$$\sqrt{mn}\sum_{j=1}^{\infty}\overline{R_j(\eta)}\lambda_j(m)\lambda_j(n)\cdot\frac{p^{\#}(ir_j)\overline{q^{\#}(ir_j)}}{\langle\phi_j,\phi_j\rangle}$$

$$= \left(\delta_{m,n}\int_0^{\infty}\tilde{p}(my)\overline{q(ny)}\frac{dy}{y^2} + \sum_{\delta|\eta}\delta\cdot\mathbb{1}_{m\eta/\delta^2=n}\mathbb{1}_{\delta|m}\int_0^{\infty}p(m(\eta/\delta^2)y)\overline{q(ny)}\frac{dy}{y^2}\right) \tag{2.11.1}$$

$$+\sum_{c=1}^{\infty}S_1(m,n;c,1)\int_0^{\infty}\int_{\mathbb{R}}U_{\log\eta}\left(p\left(\frac{m}{c^2y(x^2+1)}\right)e^{-2\pi i x\left(\frac{m}{c^2y(x^2+1)}+ny\right)}\right)\overline{q(ny)}\frac{dxdy}{y} \tag{2.11.2}$$

$$+\sum_{c=1}^{\infty}\int_0^{\infty}\int_{\mathbb{R}}p\left(\frac{mny}{c^2\left(x^2+y^2\right)}\right)e^{-2\pi i x\left(\frac{\eta}{c^2(x^2+y^2)}\right)}S_{\eta^{-1}}(m,n;c,1)\,\overline{q(ny)}\,\frac{dxdy}{y}$$

$$+\eta^{-1}\cdot\sum_{c=1}^{\infty}\int_0^{\infty}\int_{\mathbb{R}}p\left(\frac{mn\eta y}{c^2\left(x^2+\eta^2y^2\right)}\right)e^{-2\pi i x\left(\frac{1}{c^2(x^2+\eta^2y^2)}\right)}S_{\eta}(m,n;c,f_{c,k,1}(x))\,\overline{q(ny)}\,\frac{dxdy}{y}, \tag{2.11.3}$$

where f is as defined in Corollary 2.9.1.

Remark 2.11.1. *We enforce the hypotheses on $|p'(y)|$ and $|q'(y)|$ to ensure the operator $U_{\log\eta}$ preserves the space of Poincaré series.*

Proof. Combine Corollary 2.9.1 and Propositions 2.7 and 2.10. □

3. Preliminaries for Bounding the Geometric Side

Let $R > 4$ be a fixed number and T a large real number tending to infinity. Let $p_{T,R,\delta}(y)$ be the inverse Bessel transform, which exists by the general theory of the Kontorovich-Lebedev transform [GKS11], since it coincides with the Bessel transform for GL(2). Set $p = q = p_{T,R,\delta}$ for a fixed δ.

Our goal in this section is to set the necessary groundwork to prove Theorem 1.1. To guide the reader through this section, we provide a summary of the key points. Fix $\alpha,\beta > 0$. We will study asymptotics as $T \to \infty$ for integrals of the form

$$\int_{C'}^{\infty} p^{(a)}_{T,R,\delta}(\alpha y)\overline{p^{(b)}_{T,R,\delta}(\beta y)}\frac{dy}{y^2} = \int_{2\pi C'}^{\infty} p_{T,R,k}((2\pi)^{-1}\alpha y)\overline{p_{T,R,k}((2\pi)^{-1}\beta y)}\frac{dy}{y^2}, \tag{3.0.1}$$

where $(a,b) = (0,0),(0,1),(1,1)$ and C' is to be chosen following the statement of Proposition 3.0.4.

The inverse Kontorovich-Lebedev transform (see [GKS11]) allows us to write

$$p_{T,R,\delta}(y) = \frac{\sqrt{y}}{\pi}\int_{\mathbb{R}} p^{\#}_{T,R,\delta}(ir)K_{ir}(2\pi y)\frac{dr}{\Gamma(ir)\Gamma(-ir)}. \tag{3.0.2}$$

and

$$p'_{T,R,\delta}(y) = \frac{1}{2\pi\sqrt{y}} \int_{\mathbb{R}} p^{\#}_{T,R,\delta}(ir) K_{ir}(2\pi y) \frac{dr}{\Gamma(ir)\Gamma(-ir)} + 2\sqrt{y} \int_{\mathbb{R}} p^{\#}_{T,R,\delta}(ir) K'_{ir}(2\pi y) \frac{dr}{\Gamma(ir)\Gamma(-ir)}. \quad (3.0.3)$$

Remark 3.0.1. *We integrate from* 2π *on the right-hand side of* (3.0.1) *because the inverse transforms* (3.0.2)–(3.0.3) *involve* $K_{ir}(2\pi y)$ *instead of* $K_{ir}(y)$.

Observe that the three integrands above are even. Moreover, we will show that the first integral of (3.0.3) will be of negligible contribution compared to (3.0.2) in the regions of integration in (3.0.1). It will then suffice to analyze the asymptotics in T for

$$\frac{2\sqrt{\alpha\beta}}{\pi} \int_{2\pi C'}^{\infty} \left(\int_0^{\infty} p^{\#}_{T,R,\delta}(ir) K^{(a)}_{ir}(y) \frac{dr}{\Gamma(ir)\Gamma(-ir)} \right) \left(\int_0^{\infty} \overline{p^{\#}_{T,R,\delta}(is)} K^{(b)}_{is}(\beta\alpha^{-1}y) \frac{ds}{\Gamma(is)\Gamma(-is)} \right) \frac{dy}{y}. \quad (3.0.4)$$

Here, $K^{(a)}_{ir}$ denotes the a^{th} derivative of K_{ir}. We will show the following

Proposition 3.1. *When* $(a,b) = (0,0)$ *or* $(1,0)$*, the quantity* (3.0.4) *is* $\ll \sqrt{\alpha\beta} \cdot T^{R+1-a/3} \log T$ *for a fixed positive constant* c_0*. If* $\delta < 1/3$*, we have* $\ll \sqrt{\alpha\beta} \cdot T^{R+1-a/3-\delta}$.

We proceed by substituting asymptotic formulas for the Bessel functions and their derivatives given in Appendix B. Once r and s are bounded away from 0, there are three types of asymptotic expansions for the K-Bessel function, presented in Appendix B. Fix a large constant $C > 1$ and fix $C' > 1$ such that $y - Cy^{1/3} \geq 1/2$ for $y \geq C'$. Then, break up the integration over r into the following intervals, whose names align with the names of the asymptotics in Appendix B:

(1) $[0, 1/2]$,
(2) $[1/2, r_0(y)]$ (i.e., the *post-turning point region*), where $r_0(y) + Cr_0(y)^{1/3} = y$.
(3) $[r_0(y), r_1(y)]$ (i.e., the *core region*), where $r_1(y) - Cr_1(y)^{1/3} = y$.
(4) $[r_1(y), \infty)$ (i.e., the *pre-turning point region*).

Break up the integration over s into the same intervals, but with y replaced by $\alpha^{-1}\beta y$. For the remainder of the paper, we refer to these intervals by the markers (1), (2), (3), and (4).

Remark 3.1.1. *The nomenclature of these regions is made to be consistent with literature on the asymptotics of Bessel functions, starting with* [Olv54b] [Olv54a][Olv58].

We will provide asymptotics and upper bounds in y and T for these four regions, depending on if the region contributes the main term of Proposition 3.1. The reader should focus on the contributions from intervals (2)–(4); it will be easy to show that the integral over the interval (1) will be exponentially small in y and hence does not contribute to the main term.

In what follows, we will freely use the notation from Appendix B. We copy what we will use for convenience: let

$$\xi(z) := \frac{2}{3}\zeta(z)^{3/2} := \begin{cases} (z^2-1)^{1/2} - \operatorname{arcsec}(z) & z \geq 1 \\ i \cdot \left((1-z^2)^{1/2} - \log\left(\frac{1-\sqrt{1-z^2}}{z} \right) \right) & 0 < z < 1 \end{cases}$$

and

$$Y(r) := r^{2/3}\zeta\left(\frac{y}{r}\right) \quad , \quad \beta(z) := (z^2-1)^{-1/2} \quad \text{and} \quad E(\beta) = \frac{1}{24}\beta(5\beta^2+3).$$

We will derive the contributions of the regions $[0, 1/2]$, post-turning point, transition, and core regions to the integral over r in (3.0.4) for $a = 0$. The contributions to the integral over s will be identical.

A few conventions before we begin: we will freely apply Stirling's asymptotic on the Gamma function with fixed real part:

$$|\Gamma(\sigma + ir)| = (2\pi)^{1/2}|r|^{\sigma-\frac{1}{2}}e^{-\pi|r|/2}(1 + O(|r|^{-1})). \tag{3.1.1}$$

When we refer to the "main contribution" of an integral, it refers to ignoring all big-Oh's in expressions of the form $1 + O(*)$ and integrating. However, terms of the form $O(*)$ are still included.

The remainder of this section is dedicated to proving asymptotics of the contributions of the regions for $a = 0, 1$. We package them into two propositions, which we prove in §3.1–3.2. It will be convenient to let

$$G := T^{\delta}, \qquad \gamma := \frac{y - T}{G}, \qquad \delta_T := \frac{T^{1/3}}{G}.$$

Then, we have

Proposition 3.2 (Core and Post-Turning Point Region)**.** *When the relevant region has non-empty intersection with the interval $[T - cT^{\delta}, T + cT^{\delta}]$ for some positive $c > 0$, we have the following upper bounds with constants depending on C:*

- ***Post-turning region:*** *We have that*

$$y^{a+1}e^{-y}T^{R/2+2/3-a} \qquad or \qquad y^{a}(y-T)^{-1/4-a}T^{R/2+1-5a/4}$$

  depending on if $y - T \gg T^{1/3+\varepsilon}$ for some $\varepsilon > 0$ or not.
- ***Core region****. The contribution is bounded by* $\min(G, T^{1/3})T^{R/2+(2-a)/3}$.

Proposition 3.3 (Pre-Turning Region)**.** *When the pre-turning region has non-empty intersection with the interval $[T - cT^{\delta}, T + cT^{\delta}]$ for some positive $c > 0$, we have the following upper bounds with implicit constant depending on C:*

- *If $\delta < 1/3$, then, when $y \geq T-(C+1)T^{1/3}$, we have that trivial upper bound of $GT^{R/2+(2-a)/3}$. When $y \leq T + (C + 1)T^{1/3}$, we have the bounds of the post-turning region in Proposition 3.2. Otherwise, when $a = 0$ we have the upper bound*

$$M(y, T) := GT^{R/2+1}(T^2 - y^2)^{-1/4}e^{-\operatorname{arccosh}(T/y)^2G^2/4}.$$

  When $a = 1$, we have the upper bound

$$M(y, T)\left[(T^2 - y^2)^{-1} + y^{-1}\right].$$

- *If $\delta \geq 1/3$, we have the upper bound $\ll \max\left(C, \frac{T-G-y}{T^{1/3}}\right)^{-3/4} T^{R/2+1}$ when $a = 0$. When $a = 1$, one has that $\ll \max\left(C, \frac{T-G-y}{T^{1/3}}\right)^{-3/4} T^{R/2+2/3}$ when $a = 1$.*

Finally, we record the following four results for easy future reference, the last three of which may be found by Taylor expansion:

Lemma 3.4. *For the core (resp. post-turning, pre-turning) region to have non-empty intersection with $[T-cT^{\delta}, T+cT^{\delta}]$ for some constant $c > 0$, we must have that $y \sim T$ (resp. $T-y \ll CT^{\max(1/3,\delta)}$, $y - T \gg CT^{\max(1/3,\delta)}$).*

Lemma 3.5. *For $z = 1 + \varepsilon$ with $\varepsilon \ll 1$, the function $E(\beta(z))$ satisfies*

$$E(\beta(1 + \varepsilon)) \asymp \varepsilon^{-3/2} \qquad and \qquad \frac{d}{dz}E(\beta(1 + \varepsilon)) \asymp \varepsilon^{-5/2}$$

Lemma 3.6. *For $z = 1 + \varepsilon$ with $\varepsilon \ll 1$, the function $\xi(z)$ satisfies*

$$\xi(1+\varepsilon) \asymp \varepsilon^{3/2} \qquad \text{and} \qquad \xi'(1+\varepsilon) \asymp \varepsilon^{1/2}$$

Lemma 3.7. *The expression $r^{-1}E(\beta(y/r))$ is bounded by $\ll C^{-3/2}$ where the implied constant is absolute.*

3.1. **Proof of Proposition 3.2.** We start by bounding away the core region, i.e., Here, Lemma B.2 gives that

$$e^{\pi r/2}K_{ir}(y) \ll_C r^{-1/3}.$$

Multiplying by the gamma factor and applying Stirling, we get

$$\frac{p^{\#}_{T,R,\delta}(ir)}{\Gamma(ir)\Gamma(-ir)}K_{ir}(y) \ll_C T^{R/2+2/3}e^{-(r-T)^2/G^2}.$$

Integrating along the part of the core region that intersects the support of the Gaussian, which possibly is empty, yields that

$$M(y) \ll_C G\,T^{R/2+2/3},$$

which is the asserted bound for $a = 0$ in the core region. The bound for $a = 1$ follows the exact same argument and results in $\ll_C GT^{R/2+1/3}$.

Let us now proceed to the post-turning point region when $a = 0$. By substituting the asymptotics of Lemma B.1 for $K_{ir}(y)$ and bounding the factor $e^{-E(\beta(y/r))/r}$ using Lemma 3.5, we obtain one of the following:

$$|M(y)| \ll T^{\frac{R+1}{2}} \int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2}\left(\frac{y}{r}-1\right)^{-1/4} e^{-r\xi(y/r)} \tag{3.7.1}$$

when $y - T \gg T^{1/3+\varepsilon}$ for some $\varepsilon > 0$ not depending on T, or

$$|M(y)| \ll T^{R/2+1}\int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2}\,(y-r)^{-1/4}\,e^{-r\xi(y/r)} \ll T^{R/2+1}(y-T)^{-1/4}\int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2}e^{-r\xi(y/r)} \tag{3.7.2}$$

otherwise. Upon using the fact that $\left|\frac{y}{r}-1\right| \geq Cr^{-2/3}$, we obtain that $\left(\frac{y}{r}-1\right)^{-1/4} \ll T^{1/6}$ whence the first expression is

$$\ll T^{R/2+2/3}\int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2}e^{-r\xi(y/r)}dr. \tag{3.7.3}$$

Now, the expression $e^{-r\xi(y/r)}$ is majorized at the point where $\xi(y/r)$ is at a minimum. For (3.7.1) (resp. (3.7.2)), since $\left|\frac{y}{r}-1\right| \geq Cr^{-2/3}$, by Lemma 3.6 we have that $e^{-r\xi(y/r)}$ is bounded above by e^{-y} (resp. 1). Using this bounding the integral of (3.7.3) by the length of the integral and (3.7.2) by the length of the Gaussian, we obtain the proposition in the $a = 0$ case.

The proof of the $a = 1$ case is similar. After substituting asymptotics and applying Lemmas 3.5 and 3.6, when $y \asymp T$ we obtain, for $y \gg T^{1+\varepsilon}$, that

$$|M(y)| \ll T^{\frac{R-1}{2}}e^{-y}\int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2}\left[\frac{y}{r}\left(\left(\frac{y}{r}\right)^2-1\right)^{-5/4}+\xi'(y/r)\left(\left(\frac{y}{r}\right)^2-1\right)^{-1/4}\right]dr$$

$$\ll T^{\frac{R-1}{2}} \int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2}(yT^{-1/6}+T^{-1/6})dr \ll y^2 e^{-y} T^{R/2-1/3}.$$

If $y \asymp T$, then one can verify that $\xi'(y/r)\left(\left(\frac{y}{r}\right)^2-1\right)^{-1/4} \ll 1$ and so

$$\begin{aligned}|M(y)| &\ll T^{\frac{R-1}{2}} \int_{1/2}^{r_0(y)} e^{-(r-T)^2/G^2} \frac{y}{r}\left(\left(\frac{y}{r}\right)^2-1\right)^{-5/4} dr \\ &\ll y(yT^{-1}-1)^{-5/4} G T^{\frac{R-3}{2}} = y(y-T)^{-5/4} G T^{R/2-1/4}\end{aligned}$$

as desired.

3.2. **Proof of Proposition 3.3.** We will compute the asymptotics more carefully in this section so as to preserve oscillatory terms in Lemma B.3. In particular, we invoke Lemma B.4, which justifies any quantities to which we claim that $M(y)$ is asymptotic.

We start with the case that $\delta < 1/3$. Inside this case, start with the case $a = 0$. Substitute the asymptotics of Lemma B.3 to obtain that

$$M(y) \asymp T^{\frac{R+1}{2}} \int_{r_1(y)}^{\infty} e^{-(r-T)^2/G^2} S(y/r)\left(1-\left(\frac{y}{r}\right)^2\right)^{-1/4} dr$$

where $S(z) = \sin\left(\frac{\pi}{4} + r^{-1}E(\beta(z)) + r\xi(z)\right)$. Let $D = T^2 - y^2 = d(T+y)$ where $d = T - y$. Then, the Taylor expansion of $(r^2-y^2)^{-1/4}$ around T is given up to first order by

$$(r^2-y^2)^{-1/4} = D^{-1/4}\left[1 + \frac{2Th+h^2}{4D} + O\left(\left(\frac{2Th+h^2}{4D}\right)^2\right)\right].$$

But note that we only care about $|h| \ll G$ and $D \geq dT$ which, by Lemma 3.4, implies that $D \gg T^{4/3}$ while $2Th + h^2 \ll T^{4/3-\varepsilon}$. Thus,

$$M(y) \asymp T^{\frac{R+1}{2}}(1-y^2T^{-2})^{-1/4} \int_{r_1(y)}^{\infty} e^{-(r-T)^2/G^2} S(y/r)dr. \tag{3.7.4}$$

Since we are assuming that $y \leq T - (C+1)T^{1/3}$, then $r_1(y) \leq T - T^{1/3}$ and hence we can extend the r-integral to $-\infty$. Using Lemma 3.7 with large C, upon allowing $H(y/r) = i^{-1}\xi(y/r)$ we know that

$$M(y) \asymp T^{R/2+1}(T^2-y^2)^{-1/4} \int_{\mathbb{R}} e^{-(r-T)^2/G^2}\left[c_0\cos(rH(y/r)) + c_1\sin(rH(y/r))\right] dr \tag{3.7.5}$$

for some constants c_0, c_1 such that $|c_j - 2^{-1/2}| \leq C^{-3/2}$ for $j = 0, 1$. Then, the Taylor expansion of the function $rH(y/r)$ as a function of $r = T + h$ around T is

$$TH(y/T) + \operatorname{arccosh}(T/y)h + O\left(h^2 \sup_{t\in[T,T+h]}(t^2-y^2)^{-1/2}\right).$$

Again, we only care about $|h| \le T^{1/3}$ and so the error term is $O\left(\frac{G^2}{\sqrt{T^2-y^2}}\right) = O(T^{-\delta})$ since $y - T \ll CT^{1/3}$ in this situation. This term contributes negligibly to the arguments of sine and cosine in (3.7.5) and so we have that

$$M(y) \asymp (c_1\mathfrak{I} + c_0\mathfrak{R})\left[e^{iT\cdot H(y/T)}\int_{\mathbb{R}} e^{-(r-T)^2/G^2}e^{i\cdot \operatorname{arccosh}(T/y)(r-T)}dr\right].$$

We recognize the integral as none other than the Fourier transform of a Gaussian! This allows us to conclude that

$$M(y) \asymp \sqrt{\pi}\cdot G(c_1\sin(T\cdot H(y/T) + c_0\cos(T\cdot H(y/T))e^{-\operatorname{arccosh}(T/y)^2G^2/4}$$

and the upper bound when $a = 0$ and $\delta < 1/3$. The case of $a = 1$ follows similarly once we see that the contribution of $\left(1-\left(\frac{y}{r}\right)^2\right)^{-1/4} S(y/r)$ bounded, up to absolute constant, by

$$r^{-2}y\left(1-\left(\frac{y}{r}\right)^2\right)^{-1/4}\left[\left(1-\left(\frac{y}{r}\right)^2\right)^{-1} S(y/r) + rH'(y/r)S'(y/r)\right], \tag{3.7.6}$$

where we have used Lemma 3.5 in bounding the derivative of the argument of $S(z)$. Now, the first term in the brackets is $\ll y(T^2-y^2)^{-1}$. Using the fact that $H'(z) = z^{-1}\sqrt{1-z^2}$, we have that the second term in the brackets is $\ll y^{-1}\sqrt{T^2-y^2}$.

When $\delta \ge 1/3$, we use the arguments of the $\delta < 1/3$ case except that we cannot extend the integral over the pre-turning region to the whole real line. Our starting point is to make a change of coordinate in the integral of (3.7.5), but with the $(T^2-y^2)^{-1/4}$ brought back into the integral and replaced by $(r^2-y^2)^{-1/4}$. We let $r - y = ur^{1/3}$ upon which the region of integration is $[C,\infty)$.

We may treat $r := r_y(u)$ as an implicit function of u since C is sufficiently large. Then, the factor $(r^2-y^2)^{-1/4}dr$ transforms into

$$(r_y(u)-y^2)^{-1/4}\cdot\frac{dr_y}{du} = ur_y(u)^{4/3}(2-r_y(u)^{-2/3})\cdot\frac{dr_y}{du} = u^{-1/4}\cdot\frac{(2-ur_y(u)^{-2/3})^{-1/4}}{1-\frac{u}{3}r_y(u)^{-2/3}}$$

and we obtain that (3.7.5) is

$$\asymp T^{R/2+1}\int_C^\infty e^{-\frac{(r_y(u)-T)^2}{G^2}}u^{-1/4}\cdot\frac{(2-ur_y(u)^{-2/3})^{-1/4}}{1-\frac{u}{3}r_y(u)^{-2/3}}S_*(r_y(u))du.$$

where $S_*(r_y(u)) = c_0\cos\left(r_y(u)H(y/r_y(u))\right) + c_1\sin\left(r_y(u)H(y/r_y(u))\right)$ and substituting the relation for $r_y(u)$ gives

$$\asymp T^{R/2+1}\int_C^\infty e^{-\frac{(y-T+ur_y(u)^{1/3})^2}{G^2}}u^{-1/4}\frac{(2-ur_y(u)^{-2/3})^{-1/4}}{1-\frac{u}{3}r_y(u)^{-2/3}}S_*(r_y(u))du.$$

We will now prove that the integral is $\ll C^{-3/4}$ where the implied constant is absolute. Before proceeding, it will help to set

$$J_y(u) := \frac{(2-ur_y(u)^{-2/3})^{-1/4}}{1-\frac{u}{3}r_y(u)^{-2/3}} \quad , \quad W_y(u) := e^{-\frac{(y-T+ur_y(u)^{1/3})^2}{G^2}} \quad \text{, and} \quad \Psi_y(u) = r_y(u)H(y/r_y(u)).$$

We will also use

Lemma 3.8. *Consider the function $\Psi_y(u) = r_y(y)H(y/r_y(u))$. Then, $\Psi_y'(u) \gg u^{1/2}$.*

Proof. Start by calculating that

$$\Psi_y'(u) = \frac{r_y(u)}{1 - \frac{u}{3} r_y(u)^{-2/3}} \operatorname{arccosh}(r_y(u)/y).$$

Now, set $q(u) = u r_y(u)^{-2/3}$, upon which $\frac{r_y(u)}{y} = \frac{1}{1-q}$. We note that, for $0 < q < 1$, we have $\operatorname{arccosh}\left(\frac{1}{1-q}\right) \gg q^{1/2}$. Combined with the fact that the denominator above is $\Theta(1)$, we conclude. □

By Lemma 3.8, we have that $\Psi_y'(u)$ never vanishes and so an integration by parts gives

$$\left[\frac{u^{-1/4} J_y(u) W(u)}{\Psi_y'(u)}\right]_C^\infty - \int_C^\infty (\sin(\Psi_y(u)) - \cos(\Psi_y(u))) \cdot \frac{d}{du}\left(\frac{u^{-1/4} J_y(u) W(u)}{\Psi_y'(u)}\right) du.$$

The first term is $\ll C^{-3/4}$. For the second term, apply the product rule to $W(u)$ and $\frac{u^{-1/4} J_y(u)}{\Psi_y'(u)}$. Because $W(u)$ has bounded variation on $[C, \infty)$, the term with $W'(u)$ may be bounded by $\ll \sup_{u \geq C} \frac{u^{-1/4} J_y(u)}{\Psi_y'(u)} \ll C^{-3/4}$ again by Lemma 3.8. For the remaining term, we calculate the derivative to be

$$\frac{d}{du} \frac{u^{-1/4} J_y(u)}{\Psi_y'(u)} = u^{-7/4} \left(-\frac{3B(q)}{4} + \frac{q(1-q)B'(q)}{1 - q/3}\right),$$

where we use $q = q(u) = u r_y(u)^{-2/3}$ from Lemma 3.8. It is easily verified that the expression in parentheses is bounded for $0 < q < 1$ and so we have an upper bound by $\ll \int_C^\infty u^{-7/4} du \ll C^{-3/4}$.

We now try to increase the lower bound of the integral as much as possible. Given y, we must have that $|y + u r_y(u)^{1/3} - T| \ll G T^\varepsilon$ for some $\varepsilon > 0$. Thus, we can replace C with $\max\left(C, \frac{T - G - y}{(T-G)^{1/3}}\right)$. This concludes the proof of the $a = 0$ case. For $a = 1$, use (3.7.6) and run the exact same argument again, just with different powers of u in the integrand. There will also be negative powers of $r_y(u)$, which, since $r_y(u) \asymp T$ on the support of the Gaussian term, contributes a factor of $T^{-1/3}$ to the upper bound.

3.3. **Contributions Over** $[0, 1/2]$**.** We will need to collect two elementary results:

Lemma 3.9. *When $y > 0$, we have that $\frac{y^{2k}}{(2k)!} < e^y$. Moreover, $\int_0^\infty e^{t - y\cosh t} dt \ll e^{1-y}$.*

Lemma 3.10. *We have that $\int_0^\infty e^{t - y\cosh t} \cosh t dt = O(e^{1-y})$.*

Proof. Write $e^t = \cosh t + \sinh t$ to obtain

$$\int_0^\infty e^{t - y\cosh t} \cosh t dt = \int_0^\infty e^{-y\cosh t} \cosh^2(t) dt + \int_0^\infty e^{-y\cosh t} \sinh(t)\cosh(t).$$

Apply the identity $\cosh^2(t) = \frac{1 + \cosh(2t)}{2}$ to the first integral on the right-hand side and set $u = \cosh(t)$ in the second integral to obtain that

$$\int_0^\infty e^{t - y\cosh t} \cosh t dt = \frac{1}{2} \int_0^\infty e^{-y\cosh t} dt + \frac{1}{2} \int_0^\infty e^{-y\cosh t} \cosh 2t + \int_1^\infty u e^{-yu} du.$$

The first and third terms on the right-hand side are $O(e^{1-y})$, while the second term is $\frac{1}{2}K_{2i}(y) = \left(\frac{\pi}{8y}\right)^{1/2} e^{-y}(1 + O(y^{-1}))$, and so we are done. □

We will now prove that

Proposition 3.11. *The contribution over* $[0, 1/2]$ *in* r *is* $\ll e^{-T^{2-2\varepsilon}} e^{-y}$.

Proof. We prove this statement for the r integral, from which the result for the s-integral follows directly. First, let $a = 0$. It is well-known [DLMF, Eq. (10.32.9)] that

$$K_{ir}(y) = \int_0^\infty e^{-y\cosh t} \cosh(rt) dt. \tag{3.11.1}$$

Now, substitute the Taylor expansion $\cosh(rt) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$ into (3.11.1) and apply Lemma 3.9 to obtain an upper bound of

$$|K_{ir}(y) - K_0(y)| \ll \int_0^\infty e^{t-y\cosh t} \cosh(rt) dt,$$

and, by Lemma 3.10, that

$$K_{ir}(y) = K_0(y) + O\left(\int_0^\infty e^{t-y\cosh t} dt\right) = K_0(y) + O(e^{1-y}).$$

By [DLMF, Eq. (10.40.2)], we have that, for $y \geq 2\pi C'$, that $K_0(y) = \left(\frac{\pi}{2y}\right)^{1/2} e^{-y}(1 + O(y^{-1}))$. Thus, the contribution over $[0, 1/2]$ is equal to

$$\left(\left(\frac{\pi}{2y}\right)^{1/2} e^{-y}(1 + O(y^{-1})) + O(e^{1-y})\right) \int_0^{1/2} p^{\#}_{T,R,\delta}(ir) \frac{dr}{\Gamma(ir)\Gamma(-ir)} \ll e^{-T^{2-2\varepsilon}} e^{-y}. \tag{3.11.2}$$

Now let $a = 1$. Differentiate the expression (3.11.1) in y and apply the Taylor expansion along with Lemma 3.9 to obtain that

$$K'_{ir}(y) = K'_0(y) + O\left(\int_0^\infty e^{t-y\cosh t} \cosh t dt\right). \tag{3.11.3}$$

By differentiating under the integral sign, we obtain $K'_0(y) = -K_i(y)$. By [DLMF, Eq (10.40.2)], we have that $K_i(y) = \left(\frac{\pi}{2y}\right)^{1/2} e^{-y}(1 + O(y^{-1}))$. Thus, the contribution over $[0, 1/2]$ is given by (3.11.2). □

4. Proof of Theorem 1.1

We will use Propositions 3.2 and 3.3 from §3 to prove Proposition 3.1 and then Theorem 1.1. The main term will come from asymptotic formulas of the terms in (2.11.1), which we recall for convenience in the case when $p = q = p_{T,R,\delta}$:

$$\delta_{m,n} \int_0^\infty \tilde{p}_{T,R,\delta}(my)\overline{p_{T,R,\delta}(ny)} \frac{dy}{y^2} + \sum_{\delta|\ell} \delta \cdot \mathbb{1}_{m\ell/\delta^2=n} \mathbb{1}_{\delta|m} \int_0^\infty p_{T,R,\delta}(m(\ell/\delta^2)y)\overline{p_{T,R,\delta}(ny)} \frac{dy}{y^2}. \tag{4.0.1}$$

4.1. **The Main Terms $\frac{m\eta}{\delta^2} = n$ When $\delta \mid (m, \eta)$.** These may computed by first applying the Plancherel formula for the Bessel transform [Gol20, Proposition 9.15], upon which we obtain

$$n \int_0^\infty e^{-2(r-T)^2/T^{2\delta}} \frac{|\Gamma(\frac{2+R+2ir}{4})\Gamma(\frac{2+R-2ir}{4})|^2}{\Gamma(ir)\Gamma(-ir)}.$$

Applying the Stirling asymptotics yields

$$\frac{2n}{\pi} \int_0^\infty e^{-2(r-T)^2/T^{2\delta}} r^{R+1} dr \sim c_0 T^{R+1+\delta},$$

where c_0 is a fixed positive constant. This proves the first main term in Theorem 1.1.

4.2. **The Error Term.** We will require the following

Proposition 4.1 ([Gol20, Proposition 9.5])**.** *Let $k = 0, 1$ and $\varepsilon > 0$ and $R > 4$ with $1 \leq a < \frac{R}{4}$. Then for $y > 0$ and $T \to \infty$, we have*

$$|p^{(k)}_{T,R,\delta}(y)| \ll y^{\frac{1}{2}-k+2a} T^{\varepsilon+\frac{R+1}{2}+\delta-a}.$$

If $y > 1$, we may replace a with an arbitrarily small real number.

Proof. The case $k = 0$ is exactly [Gol20, Proposition 9.5]. For $k = 1$, the exact same argument goes through, with a factor of $\sqrt{y}$ changing into $y^{-1/2}$. □

To prove the result on the error term (2.11.3), break the y-integral into two pieces, one on $[0, 1]$ and the other on $[1, \infty)$. On $[0, 1]$, upper bound $p(y)$ by using $a = R/4$ for some $\varepsilon > 0$ and upper bound $q(y)$ by using $a = 1$ in the $k = 0$ case of Proposition 4.1. Then, apply the bounds $|S_{\frac{\eta}{\delta^2}}(m, n; c, f_{c,k,\delta})| \ll c$ and $\varphi(c) \ll c$. On $[1, \infty)$, use $a = R/4$ to bound $p(y)$ and $a = 1$ to bound $q(y)$. In all cases, the infinite sum in c is convergent. To prove the result on the error term (2.11.2), we can run the same argument, in combination with Corollary A.3.1. The total error is $O(T^{\frac{R+1}{2}+2\delta} M)$ where

$$M = \min\{(mn\eta)^{R/2+1/2+\varepsilon}, (mn)^{R/2+1/2+\varepsilon}+\eta^{R/2+1/2+\varepsilon}, (m\eta)^{R/2+1/2+\varepsilon}+n^{R/2+1/2+\varepsilon}, (n\eta)^{R/2+1/2+\varepsilon}+m^{R/2+1/2+\varepsilon}\},$$

as expected.

4.3. **The Main Term when $m = n$.** Recall that we want an upper bound for

$$\frac{2\sqrt{\alpha\beta}}{\pi} \int_{2\pi C'}^\infty \left(\int_0^\infty p^{\#}_{T,R,\delta}(ir) K^{(a)}_{ir}(y) \frac{dr}{\Gamma(ir)\Gamma(-ir)} \right) \left(\int_0^\infty \overline{p^{\#}_{T,R,\delta}(is)} K^{(b)}_{is}(\beta\alpha^{-1} y) \frac{ds}{\Gamma(is)\Gamma(-is)} \right) \frac{dy}{y}.$$

Remark 4.1.1. *The proof of the bound on the main term in* [Gol20] *uses the Plancherel theorem of the Kontorovich-Lebedev transform. Because we may have main terms when $\alpha\beta^{-1} \neq 1$, which is a similar to a problem encountered in* [GHH24]*. There, the authors integrate away the two Bessel function into a ${}_2F_1$ hypergeometric function. We cannot do this due to the $\frac{dy}{y}$ factor in* (3.0.4).

Appendix A. Properties of the Operator $U_{\log \ell}$

Lemma A.1. *Given initial data $u_0 = f$ and $\frac{d}{dt}\big|_{t=0} u_t = 0$, one has*

$$u_{\log \ell}(z) = \frac{1}{2\pi} \frac{d}{dt}\bigg|_{t=\log \ell} \int\limits_{\rho(z,z')<\log \ell} \frac{f(z')}{\sqrt{2(\cosh t - \cosh \rho(z,z')}} \frac{dx'dy'}{(y')^2}.$$

Proof. See [Hel06, Eq. 8], [Gus13, Eq. (1.21)], or [KI81]. □

Lemma A.2. *The function $U_{\log \ell}(p(my)e^{2\pi imx})$ factors as $\tilde{p}(y)e^{2\pi imx}$.*

Proof. First, import the result of Lemma A.1, so that the function is equal to

$$\frac{1}{\pi} \frac{d}{dt}\bigg|_{t=\log \ell} \int\limits_{\rho(z,z')<t} \frac{p(my')}{\sqrt{\cosh(t) - \cosh(\rho(z,z')}} \frac{dx'dy'}{(y')^2}.$$

Now, make the change of variable $x' \mapsto x + y \sinh r \cos\theta$ and $y' \mapsto y \cosh r + y \sinh r \sin\theta$. Then, the integral above becomes

$$\frac{1}{\pi} \frac{d}{dt}\bigg|_{t=\log \ell} \int\limits_0^t \int\limits_0^{2\pi} \frac{p(my\cosh r + my \sinh r \sin\theta) e^{2\pi im(x + y \sinh r \cos\theta)}}{\sqrt{\cosh t - \cosh r}} \times \frac{\sinh r d\theta dr}{\cosh r + \sinh r \sin\theta}.$$

Now, make the change of variable $u = r/t$.

$$\frac{1}{\pi} \frac{d}{dt}\bigg|_{t=\log \ell} \int\limits_0^1 \int\limits_0^{2\pi} \frac{p(my\cosh(ut) + my\sinh(ut)\sin\theta)\, e^{2\pi im\,(x + y\sinh(ut)\cos\theta)}\, \sinh(ut)}{\sqrt{\cosh t - \cosh(ut)}\,(\cosh(ut) + \sinh(ut)\sin\theta)}\, t d\theta\, du.$$

Now, the nth Fourier coefficient is given by the integral

$$\frac{1}{\pi} \int\limits_0^1 \int\limits_0^1 \int\limits_0^{2\pi} \frac{d}{dt}\bigg|_{t=\log \ell} \frac{p(my\cosh(ut) + my\sinh(ut)\sin\theta)\, e^{2\pi im\,(x + y\sinh(ut)\cos\theta)}\, \sinh(ut)}{\sqrt{\cosh t - \cosh(ut)}\,(\cosh(ut) + \sinh(ut)\sin\theta)} \times e^{-2\pi inx}\, dx d\theta\, du.$$

Collecting the x-dependencies in the above integral, we obtain $e^{2\pi i(m-n)x}$. Thus, the only Fourier mode with non-zero coefficient is the nth mode, giving the first statement of the lemma. □

Lemma A.3. *The series $\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} U_{\log r}\left(p(my)e^{2\pi im \Re(\gamma z)}\right)$ converges absolutely when $p'(y)$ decays more than quadratically towards zero and infinity. More specifically, the decay of each summand towards zero and infinity matches that of the function $p(y) + yp'(y)$.*

Proof. We give the proof of the case in the case of $\gamma = 1$. The remaining cases follow the same steps. Moreover, after we take the derivative in t, we can bring the absolute value into the integrand, at which point the argument presented below can be followed verbatim. By the work in the proof of Lemma A.2, we want to find the decay properties of

$$\left| \frac{d}{dt}\bigg|_{t=\log \ell} \int\limits_0^1 \int\limits_0^{2\pi} \frac{p(my\cosh(ut) + my\sinh(ut)\sin\theta)\, e^{2\pi im\,(x + y\sinh(ut)\cos\theta)}\, \sinh(ut)}{\sqrt{\cosh t - \cosh(ut)}\,(\cosh(ut) + \sinh(ut)\sin\theta)}\, d\theta\, du \right|. \tag{A.3.1}$$

Let us break the integrand into two parts. Define

$$A(x,y,u,t,\theta) := \frac{te^{2\pi im(x+y\sinh(ut)\cos\theta)}\sinh(ut)}{\sqrt{\cosh t-\cosh(ut)}(\cosh(ut)+\sinh(ut)\sin\theta)}.$$

Then, the quantity above is bounded above by

$$\begin{aligned}\int_0^1\int_0^{2\pi} &\left|p\Big(my\cosh(u\log\ell)+my\sinh(u\log\ell)\sin\theta\Big)\partial_t|_{t=\log\ell}A\right| \\ &+\left|mu\cdot p'\Big(y\cosh(u\log\ell)+y\sinh(u\log\ell)\sin\theta\Big)\Big(y\sinh(u\log\ell)\right. \\ &\left.+y\cosh(u\log\ell)\sin\theta\Big)A(x,y,u,\log\ell,\theta)\right|d\theta\,du\end{aligned} \tag{A.3.2}$$

For y large, recall that $p(y)\ll y^{-C}$, for $C>1$. Thus, the function p in the integrand will be bounded by its value on the smallest y-coordinate in the region of integration, i.e., $\ll (my/\ell)^{-C}\ll y^{-C}$. Similarly, for y close to 0, the function p will be bounded by $\ll (my\ell)^{1+\varepsilon}\ll y^{1+\varepsilon}$. Then, the quantity above decays like

$$\ll y^{-C}\int_0^1\int_0^{2\pi}|A(x,y,u,\log\ell,\theta)|+|\partial_t|_{t=\log\ell}A(x,y,u,t,\theta)|dud\theta$$

as $y\to\infty$ and

$$\ll y^{1+\varepsilon}\int_0^1\int_0^{2\pi}|A(x,y,u,\log\ell,\theta)|+|\partial_t|_{t=\log\ell}A(x,y,u,t,\theta)|dud\theta$$

as $y\to 0$.

For the contribution from the term involving the derivative of p, we can repeat the above argument for $y\cdot\partial_t p$ with C replaced by $C+1$ to conclude the proof. □

Corollary A.3.1. *The function $\tilde{p}(y)$ from Lemma A.2 decays like $p(y)+yp'(y)$.*

Proof. The expression (A.3.1) is, by Lemma A.2, precisely $|\tilde{p}(y)|$. □

We also record another consequence of this proof that we will use later.

Corollary A.3.2. *The decay of $U_{\log\ell}(p(y)e^{-2\pi if(x,y)})$ is that of the function $p(y)+yp'(y)$ when f is a real-valued function.*

Proof. The function will be bounded above by (A.3.1) with the exponential term replaced by

$$e^{2\pi imf(x+y\sinh(ut)\cos\theta,y\cosh(ut)+y\sinh(ut)\sin\theta)},$$

which is bounded. □

Appendix B. Asymptotics on Bessel Functions

We will now provide a comprehensive list of all large imaginary order asymptotics of Bessel functions. It is by no means a complete list of all known asymptotics in the order – for instance [Olv97] has many asymptotics for large real order – but will suffice for our purposes. It will be convenient to set

$$\xi(z) := \frac{2}{3}\zeta(z)^{3/2} := \begin{cases} (z^2-1)^{1/2} - \operatorname{arcsec}(z) & z \geq 1 \\ i \cdot \left((1-z^2)^{1/2} - \log\left(\frac{1-\sqrt{1-z^2}}{z}\right)\right) & 0 < z < 1 \end{cases}$$

and $Y(r) := r^{2/3}\zeta\left(\frac{y}{r}\right)$. We will also need the special functions $\beta(z) = (z^2-1)^{-1/2}$ and $E(\beta) = \frac{1}{24}\beta(5\beta^2+3)$. One can readily verify that

$$\xi'(z) = \begin{cases} \frac{\sqrt{z^2-1}}{z} & z \geq 1+\delta \\ -\frac{\sqrt{1-z^2}}{z} & 0 < z \leq 1-\delta \end{cases} \tag{B.0.1}$$

Remark B.0.1. *For example, by* [Dun24, (2.8)]*, one can write $\xi = \beta^{-1} - \operatorname{arccot}(\beta)$, though we do not use this formula.*

Lemma B.1. *The Bessel function $K_{ir}(y)$ enjoys the following asymptotic formulae in the post-turning point region.*

$$K_{ir}(y) = \left(\frac{\pi}{2r}\right)^{1/2} e^{-\pi r/2}\left(\left(\frac{y}{r}\right)^2 - 1\right)^{-1/4} e^{-r\xi\left(\frac{y}{r}\right) - \frac{E\left(\beta\left(\frac{y}{r}\right)\right)}{r}}(1 + O(r^{-2}\xi(y/r)^{-2}))$$

and

$$\begin{aligned} r \cdot K'_{ir}(y) = &\left(\frac{\pi}{2r}\right)^{1/2} e^{-\pi r/2} e^{-r\xi\left(\frac{y}{r}\right) - \frac{E(\beta)}{r}}\left[\frac{y}{2r}\left(\left(\frac{y}{r}\right)^2 - 1\right)^{-5/4} + \left(\left(\frac{y}{r}\right)^2 - 1\right)^{-1/4}\left(r\xi'\left(\frac{y}{r}\right) - \frac{(E\circ\beta)'\left(\frac{y}{r}\right)}{r}\right)\right] \\ &\times (1 + O(r^{-2}\xi(y/r)^{-2})) \\ &+ O(r^{-1}\xi(y/r)^{-2})\left(\frac{\pi}{2r}\right)^{1/2} e^{-\pi r/2}\left(\left(\frac{y}{r}\right)^2 - 1\right)^{-1/4} e^{-r\xi\left(\frac{y}{r}\right) - \frac{E\left(\beta\left(\frac{y}{r}\right)\right)}{r}}. \end{aligned}$$

Proof. The first equation is a simplified form of [Dun24, Theorem 2.1] combined with the observation from Remark B.0.1, which says that $\xi \sim \frac{1}{3\beta^3}$. Then, asymptotic analysis of the error term of [Dun24, Theorem 2.1] yields the error in the first result above. The second is derived from the same result and [Dun20, Theorem 1.1]. □

Lemma B.2. *Assume $y > 1$. In the core region, the Bessel function $K_{ir}(y)$ enjoys the following asymptotic formulas:*

$$K_{ir}(y) = \pi\sqrt{2}r^{-1/3}e^{-\pi r/2}\Theta(1)\left(\Theta(1) + O(r^{-4/3})\right)$$

and

$$K'_{ir}(y) = \pi\sqrt{2}e^{-\pi r/2}r^{-2/3}\Theta(1)\left(\Theta(1) + O(r^{-2/3})\right),$$

where the $\Theta(1)$ terms depend on the global constant C of §3.

Proof. We may apply [Bal67, Eq. (20)] with the observation that ξ is bounded in the transition and core regions to obtain that

$$K_{ir}(y) = \pi\sqrt{2}r^{-1/3}e^{-\pi r/2}g\left(\frac{y}{r}\right)\left(\operatorname{Ai}(Y)(1 + O(r^{-2})) + \frac{\operatorname{Ai}'(Y)}{r^{4/3}} + O(r^{-3})\right). \tag{B.2.1}$$

and

$$K'_{ir}(y) = \pi\sqrt{2}\cdot r^{-2/3}e^{-\pi r/2}g\left(\frac{y}{r}\right)^{-1}\left(\mathrm{Ai}(Y)O(r^{-2/3}) + \mathrm{Ai}'(Y)(\Theta(1)+O(r^{-2})) + O(r^{-2/3})\right), \quad \text{(B.2.2)}$$

where Ai is the Airy function and $g(z)$ is as defined in the lemma statement.

In the core region, the hypotheses of the theorem force $Y < 1+\delta$, and hence the contribution of both Ai and Ai′ is $\Theta(1)$. This implies that $\xi \ll y^{-2/3}$, and since $y > 1$, the contribution of $g\left(\frac{y}{r}\right)$ is also $\Theta(1)$. Combining everything yields claim for $K_{ir}(y)$ in the core region. The claim about $K'_{ir}(y)$ follows similarly. □

Lemma B.3. *The Bessel function $K_{ir}(y)$ enjoys the following asymptotic formulae in the pre-turning point region:*

$$\frac{-c_0K_{ir}(y)}{\pi} = S\left(\frac{y}{r}\right)(2\pi r)^{-1/2}e^{-\pi r/2}\left(1-\left(\frac{y}{r}\right)^2\right)^{-1/4}\left(1+O(r^{-2}\xi(y/r)^{-2})\right)$$

and

$$\begin{aligned}\frac{-c_0rK'_{ir}(y)}{\pi} =& (2\pi r)^{-1/2}e^{-\pi r/2}\left[\frac{y}{2r}\left(1-\left(\frac{y}{r}\right)^2\right)^{-5/4}S\left(\frac{y}{r}\right)+\left(1-\left(\frac{y}{r}\right)^2\right)^{-1/4}S'\left(\frac{y}{r}\right)\right](1+O(r^{-2}\xi(y/r)^{-2}))\\ &+O(r^{-1}\xi(y/r)^{-2})\cdot(2\pi r)^{-1/2}e^{\pi r/2}\left(1-\left(\frac{y}{r}\right)^2\right)^{-1/4}S\left(\frac{y}{r}\right),\end{aligned}$$

where c_0 is a fixed non-zero constant and $S\left(\frac{y}{r}\right) = \sin\left(\frac{\pi}{4}+r^{-1}E\left(\beta\left(\frac{y}{r}\right)\right)+r\xi\left(\frac{y}{r}\right)\right)$.

Proof. By the connection formula [Dun24, (4.10)], we have that

$$\frac{\sinh(\pi r)}{\pi}K_{ir}(y) = -\Im(I_{ir}(y)),$$

where I_{ir} is the I-Bessel function. Applying [Dun24, Theorem 2.1], we obtain the first claim. To get the second, directly differeniate the formula for $I_{ir}(rz)$ given in *loc. cit.* and take its imaginary part. □

Lemma B.4. *The error terms in the asymptotics of Lemma B.3 are of finite variation not depending on y.*

Proof. By the Cauchy integral formula and the decay rates of the error terms, we obtain an upper bound for the finite variation of these quantities independent of y. □

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