

Norm residue isomorphism theorem: more cohomology

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1. INTRODUCTION

In order to prove the equivalence of the various conditions $H90(n)$, $BK(n)$, and $BL(n)$, we'll need to pass through some more complicated cohomology groups. In particular, we'll need to be able to extend our cohomology theories to singular varieties; and we'll need to be able to talk about cohomology with supports (for both smooth and singular varieties). We'll introduce these notions today, and prove some results which will be useful next time.

2. COHOMOLOGY OF SINGULAR VARIETIES

For each nonnegative integer m , we have a scheme $\Delta^m = \text{Spec } k[t_0, \dots, t_m]/(1-t_0-\dots-t_m)$. These are equipped with “coface” maps $\partial_i : \Delta^{m-1} \rightarrow \Delta^m$ defined by sending t_i to 0, and codegeneracy maps $\Delta^m \rightarrow \Delta^{m-1}$ sending $t_i \mapsto t_i + t_{i+1}$, and so fit together to form a cosimplicial scheme Δ^\bullet . We write $\partial\Delta^m$ for the union of the $\partial_i\Delta^m$. There are $m+1$ maps $\Delta^0 \simeq \text{Spec } k \rightarrow \Delta^m$, for each $0 \leq i \leq m$ sending $t_i \mapsto 1$ and all other $t_j \mapsto 0$; these embeddings of Δ^0 in Δ^m are called the vertices of Δ^m .

Note that this is a highly singular variety: whereas previously we've done everything in the category of smooth schemes over a field, we now want to be able to take the cohomology of sheaves defined a priori only on this category with respect to singular schemes.

Let X be any scheme over a field k . This has a contravariant functor associated to it, namely $U \mapsto X(U) = \text{Hom}_{\mathbf{Sch}/k}(U, X)$, which is a sheaf of sets. To make this a sheaf of abelian groups, the natural thing to do is to take linear combinations: $U \mapsto \mathbb{Z}[\text{Hom}_{\mathbf{Sch}/k}(U, X)]$. This is not necessarily a sheaf of abelian groups for any given topology, but we can sheafify it: in particular, we define $\mathbb{Z}[X]$ to be the Nisnevich sheafification of $U \mapsto \mathbb{Z}[\text{Hom}_{\mathbf{Sch}/k}(U, X)]$, the “natural” Nisnevich sheaf of abelian groups associated to X . This is a functor $(\mathbf{Sch}/k)^{\text{op}} \rightarrow \mathbf{Ab}$; we denote by $\mathbb{Z}_{\mathbf{Sm}}[X]$ its restriction to $(\mathbf{Sm}/k)^{\text{op}}$.

For any scheme X of finite type over k and complex \mathcal{F} of Nisnevich sheaves on \mathbf{Sm}/k , we define $H^p(X_{\mathbf{Sm}}, \mathcal{F}) = \text{Hom}_{D(\mathbf{Sm}/k)}(\mathbb{Z}_{\mathbf{Sm}}[X], \mathcal{F}[p])$. If X is smooth, then $\mathbb{Z}_{\mathbf{Sm}}[X]$ as a sheaf on \mathbf{Sm}/k is generated by the representable sheaf $\text{Hom}(-, X)$ and so this is the usual Nisnevich cohomology.

We might hope that analogously if X is singular and \mathcal{F} is actually the restriction to \mathbf{Sm}/k of a complex of sheaves on \mathbf{Sch}/k , then $H^p(X_{\mathbf{Sm}}, \mathcal{F}) = H^p(X, \mathcal{F}')$. However, this is not in general true; in good cases it turns out to instead be equal to the cohomology in a different Grothendieck topology, namely the cdh topology, which we will not get into now (if ever).

We want to establish a version of Mayer-Vietoris. To start with, suppose that $X = X_1 \cup X_2$ is the union of closed subschemes. Then we have an exact sequence of sheaves

$$0 \rightarrow \mathbb{Z}_{\mathbf{Sm}}[X_1 \cap X_2] \rightarrow \mathbb{Z}_{\mathbf{Sm}}[X_1] \times \mathbb{Z}_{\mathbf{Sm}}[X_2] \rightarrow \mathbb{Z}_{\mathbf{Sm}}[X] \rightarrow 0$$

and taking $R\text{Hom}(-, \mathcal{F})$ we get a Mayer-Vietoris exact sequence.

More generally, if $X = \cup X_i$ is the union of finitely many closed subschemes X_i , then we have the Čech spectral sequence

$$E_1^{p,q} = \check{C}^p(X, H^q(-_{\mathbf{Sm}}, \mathcal{F})) = \prod_{i_0 < \dots < i_p} H^q((\cap X_{i_r})_{\mathbf{Sm}}, \mathcal{F}) \implies H^{p+q}(X_{\mathbf{Sm}}, \mathcal{F}).$$

If X and all of the X_i and their intersections are smooth, then the Čech cohomology agrees with the Nisnevich cohomology $H_{\text{nis}}^p(X, \mathcal{F})$.

Let's apply this spectral sequence to compute the cohomology of $\partial\Delta^m$. Set $X_i = \partial_i\Delta^m$, and recall that $\partial\Delta^m = \cup X_i$ by definition. Suppose that \mathcal{F} is a complex of sheaves on \mathbf{Sm}/k with homotopy-invariant cohomology. Then since every nonempty intersection of the X_i is homotopic to $\Delta^0 \simeq \text{Spec } k$, each $H^q((\cap X_{i_r})_{\mathbf{Sm}}, \mathcal{F})$ is equal to $H^q(k, \mathcal{F})$ if the intersection is nonempty. If $p \geq m$, the only possible intersection is of all of the X_i , which is empty, and so $E_1^{p,q}$ vanishes for $p \geq m$. The differential $d_1 : E_1^{p,q} \rightarrow E_1^{p+1,q}$ is exact except when $p = 0$ or $p = m - 1$, in which case its homology is one-dimensional, so that $E_2^{p,q} = H^p(S^{m-1}, H^q(k, \mathcal{F}))$, where the spectral sequence degenerates. Therefore $H^p((\partial\Delta^m)_{\mathbf{Sm}}, \mathcal{F}) = H^p(k, \mathcal{F}) \oplus H^{p+1-m}(k, \mathcal{F})$.

Since each $\mathbb{Z}[X]$ is in fact also an étale sheaf, we can work as above in the étale topology. (We'll write $\mathbb{Z}_{\mathbf{Sm}}[X]$ to denote the restriction to $(\mathbf{Sm}/k)_{\text{ét}}$ by abuse of notation.) For the étale topology, the failure of the cohomology to play well with restriction to smooth schemes is remedied at least in certain cases.

Let $\pi : \mathbf{Sm}_{\text{ét}} \rightarrow \mathbf{Sm}_{\text{nis}}$ be the change-of-topology morphism. For a complex of étale sheaves \mathcal{F} on \mathbf{Sch}/k , let $\mathcal{F}_{\mathbf{Sm}}$ be its restriction to $(\mathbf{Sm}/k)_{\text{ét}}$, and $R\pi_*\mathcal{F}_{\mathbf{Sm}}$ be the resulting complex of Nisnevich sheaves on $(\mathbf{Sm}/k)_{\text{nis}}$, whose cohomology is taken in the sense defined above.

Proposition 1. *Suppose that $X = \cup X_i$ is a finite union of smooth closed subschemes, whose intersections are all smooth. If \mathcal{F} is a complex of étale sheaves with cohomology locally constant torsion prime to the characteristic of k , then*

$$H_{\text{ét}}^*(X, \mathcal{F}) \simeq H_{\text{ét}}^*(X_{\mathbf{Sm}}, \mathcal{F}_{\mathbf{Sm}}) \simeq H_{\text{nis}}^*(X_{\mathbf{Sm}}, R\pi_*\mathcal{F}_{\mathbf{Sm}}).$$

This will follow from the following lemma. Write $\check{C}(X)$ for the Čech complex of X .

Lemma 2. *Suppose that $X = \cup X_i$ is a finite union of closed subschemes X_i , and \mathcal{F} is a complex of étale sheaves with locally constant torsion cohomology on \mathbf{Sch}/k , with torsion prime to the characteristic of k . Then there is a natural quasi-isomorphism*

$$R\text{Hom}_{\mathbf{Sch}_{\text{ét}}}(\mathbb{Z}[X], \mathcal{F}) \rightarrow R\text{Hom}_{\mathbf{Sch}_{\text{ét}}}(\mathbb{Z}[\check{C}(X)], \mathcal{F}).$$

Proof. It suffices to take $X = X_1 \cup X_2$, and conclude from there by induction. Let $p : X \rightarrow k$ be the structure map, so that $Rp_*\mathcal{F} = R\text{Hom}_{\mathbf{Sch}}(\mathbb{Z}[X], \mathcal{F})$. If $\iota^i : X_i \hookrightarrow X$ are the inclusions, then $Rp_*R\iota_*^i\mathcal{F} = R\text{Hom}(\mathbb{Z}[X_i], \mathcal{F})$, and similarly for the inclusion $\iota^{12} : X_1 \cap X_2 \rightarrow X$. The triangle $\mathcal{F} \rightarrow \iota_*^1\mathcal{F} \oplus \iota_*^2\mathcal{F} \rightarrow \iota_*^{12}\mathcal{F} \rightarrow$ induces (via p_* and the above identities) the corresponding triangle for $R\text{Hom}_{\mathbf{Sch}_{\text{ét}}}(-, \mathcal{F})$, which is the defining property of the Čech complex in the derived category; therefore they are identified up to quasi-isomorphism. \square

Proof of Proposition 1. By Lemma 2,

$$H_{\text{ét}}^p(X, \mathcal{F}) = R\text{Hom}_{\mathbf{Sch}_{\text{ét}}}(\mathbb{Z}[X], \mathcal{F}[p]) = R\text{Hom}_{\mathbf{Sch}_{\text{ét}}}(\mathbb{Z}[\check{C}(X)], \mathcal{F}[p]).$$

Since the X_i and their intersections are smooth, we can restrict the Čech complex to the smooth category, where the étale cohomology and the Nisnevich cohomology agree, so that this is equal to

$$R\text{Hom}_{\mathbf{Sm}_{\text{ét}}}(\mathbb{Z}[\check{C}(X)], \mathcal{F}_{\mathbf{Sm}}[p]) = R\text{Hom}_{\mathbf{Sm}_{\text{nis}}}(\mathbb{Z}[\check{C}(X)], R\pi_*\mathcal{F}[p]).$$

But Mayer-Vietoris in the smooth setting is essentially the statement $\mathbb{Z}[\check{C}(X)] \simeq \mathbb{Z}_{\mathbf{Sm}}[X]$, and so the left-hand side is just $H_{\text{ét}}^p(X_{\mathbf{Sm}}, \mathcal{F}_{\mathbf{Sm}})$, while the right-hand side is $H_{\text{nis}}^p(X_{\mathbf{Sm}}, R\pi_*\mathcal{F}_{\mathbf{Sm}})$. \square

Question for the audience: how are the hypotheses on the sheaves used?

We can now apply this in a more familiar setting. Recall that the sheaf $L/\ell(n)$ is the truncation $\tau^{\leq n}R\pi'_*\mathbb{Z}/\ell(n)$, where we write π' for the change-of-topology map $(\mathbf{Sm}/k)_{\text{ét}} \rightarrow (\mathbf{Sm}/k)_{\text{Zar}}$.

Corollary 3. *Let $X = \cup X_i$ as above. Then for any $p \leq n$ there is a natural isomorphism*

$$H^p(X_{\mathbf{Sm}}, L/\ell(n)) \simeq H_{\text{ét}}^p(X, \mu_{\ell}^{\otimes n}).$$

Proof. If we could show that $H^p(X_{\mathbf{Sm}}, \tau^{\leq n}\mathcal{F}) \simeq H^p(X_{\mathbf{Sm}}, \mathcal{F})$ for every complex \mathcal{F} of Nisnevich sheaves, we would be done by taking $\mathcal{F} = R\pi_*\mathbb{Z}/\ell(n)$, since $H^p(X_{\mathbf{Sm}}, R\pi_*\mathbb{Z}/\ell(n)) = H_{\text{ét}}^p(X, \mu_{\ell}^{\otimes n})$ as in the introduction. But this is true almost by definition: we can formalize it by decomposing \mathcal{F} into its truncation $\leq n$ and $> n$, and choose $\tau^{> n}\mathcal{F}$ to be a complex of injective degrees vanishing in degrees at most n to verify that this notion of cohomology respects these truncations. \square

The hypotheses on X are not necessary; the more general statements for all $X \in \mathbf{Sch}/k$ are due to Suslin, using alterations and the h -topology.

The final thing in this section is a rather technical proposition which Haodong will need eventually and so I'll state:

Proposition 4. *Let W be a smooth semilocal scheme with a subscheme $X \subseteq W$ which is a union of smooth closed subschemes $X_i \subseteq W$ all of whose intersections are smooth. If \mathcal{F} is a complex of sheaves with transfers with homotopy-invariant cohomology, then $H^q(X_{\mathbf{Sm}}, \mathcal{F})$ is isomorphic to the cohomology of the total complex of the double Čech complex $\check{C}(X, \mathcal{F})$.*

Proof: exercise: read Haesemeyer and Weibel.

3. COHOMOLOGY WITH SUPPORTS

In this section we introduce a notion of cohomology with supports, and show that assuming $\text{BL}(n-1)$, the Beilinson-Lichtenbaum map α at least induces isomorphisms on the cohomology with supports when the support is not too big.

Let X be a smooth scheme over k , and Z be a closed subspace of X . Then $\mathbb{Z}[X - Z]$ is a subsheaf of $\mathbb{Z}[X]$, and we define $\mathbb{Z}_Z[X] = \mathbb{Z}[X]/\mathbb{Z}[X - Z]$. For \mathcal{F} a complex of sheaves as usual, its cohomology with supports is

$$H_Z^p(X, \mathcal{F}) = \mathrm{Hom}_{D(\mathbf{Sm}/k)}(\mathbb{Z}_Z[X], \mathcal{F}[p]).$$

For an intermediate subspace $Z \subset Y \subset X$, we have an exact sequence

$$0 \rightarrow \mathbb{Z}_{Y-Z}[X - Z] \rightarrow \mathbb{Z}_Y[X] \rightarrow \mathbb{Z}_Z[X] \rightarrow 0,$$

as can be seen easily by expanding the definitions. There is a corresponding long exact sequence for the cohomology with supports, which is natural in \mathcal{F} .

Let $D(\mathbf{Cor}/k)$ be the derived category of sheaves with transfers, and recall its full triangulated subcategory $\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}$. The functor $D(\mathbf{Cor}/k) \rightarrow D(\mathbf{Sm}/k)$ induced by forgetting the transfer structures is triangulated, and so every triangle in $\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}$ induces a triangle in $D(\mathbf{Sm}/k)$. We apply this to prove the following lemma.

Lemma 5. *Suppose that X is smooth and Z is a smooth closed subscheme of codimension c , and \mathcal{F} is a complex of sheaves with transfers with homotopy-invariant cohomology. Then there is a canonical isomorphism*

$$H^{p-c}(Z, \mathcal{F}_{-c}) \simeq H_Z^p(X, \mathcal{F}).$$

Here \mathcal{F}_{-c} is the c th iteration of the $\mathcal{F} \mapsto \mathcal{F}_{-1}$ construction from Hung's talk.

Proof. We can assume that \mathcal{F} is bounded below, since if not we can replace it with a truncation without changing the relevant cohomology groups. There is a Gysin triangle $C_*\mathbb{Z}_{\mathrm{tr}}(X - Z) \rightarrow C_*\mathbb{Z}_{\mathrm{tr}}(X) \rightarrow C_*\mathbb{Z}_{\mathrm{tr}}(Z) \otimes \mathbb{L}^c = C_*\mathbb{Z}_{\mathrm{tr}}(Z)(c)[2c] \rightarrow$ in $\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}$, where $\mathbb{L} = \mathbb{Z}(1)[2]$. We get an induced triangle in $D(\mathbf{Sm}/k)$, where we also have a triangle $\mathbb{Z}[X - Z] \rightarrow \mathbb{Z}[X] \rightarrow \mathbb{Z}_Z[X] \rightarrow$ defining $\mathbb{Z}_Z[X]$. The map $\mathbb{Z}[X] \rightarrow C_*\mathbb{Z}_{\mathrm{tr}}(X)$ extends to this triangle, yielding a canonical map $\mathbb{Z}_Z[X] \rightarrow C_*\mathbb{Z}_{\mathrm{tr}}(Z) \otimes \mathbb{L}^c$. Taking $\mathrm{Hom}(-, \mathcal{F}[p])$ gives a morphism of long exact sequences. In fact

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}}(C_*\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{F}[p]) \simeq \mathrm{Hom}_{D(\mathbf{Sm}/k)}(\mathbb{Z}[X], \mathcal{F}[p]) = H^p(X, \mathcal{F})$$

and similarly for $X - Z$, and so by the 5-lemma the canonical map $\mathbb{Z}_Z[X] \rightarrow C_*\mathbb{Z}_{\mathrm{tr}}(Z) \otimes \mathbb{L}^c$ yields an isomorphism on cohomology

$$H_Z^p(X, \mathcal{F}) = \mathrm{Hom}_{D(\mathbf{Sm}/k)}(\mathbb{Z}_Z[X], \mathcal{F}[p]) \simeq \mathrm{Hom}_{\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}}(C_*\mathbb{Z}_{\mathrm{tr}}(Z) \otimes \mathbb{L}^c, \mathcal{F}[p]).$$

Replacing \mathcal{F} by $\mathcal{F}_{-c} = R\mathrm{Hom}(\mathbb{L}^c, \mathcal{F})[c]$ gives

$$H^{p-c}(Z, \mathcal{F}_{-c}) = \mathrm{Hom}_{\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}}(C_*\mathbb{Z}_{\mathrm{tr}}(Z), R\mathrm{Hom}(\mathbb{L}^c, \mathcal{F})[p]).$$

By the Hom-tensor adjunction, this is canonically isomorphic to

$$\mathrm{Hom}_{\mathbf{DM}_{\mathrm{nis}}^{\mathrm{eff}}}(C_*\mathbb{Z}_{\mathrm{tr}}(Z) \otimes \mathbb{L}^c, \mathcal{F}[p]),$$

which we just proved was canonically isomorphic to $H_Z^p(X, \mathcal{F})$. □

Recall from Hung's talk that $\mathbb{Z}/\ell(n)_{-1} \simeq \mathbb{Z}/\ell(n-1)[-1]$, so that $\mathbb{Z}/\ell(n)_{-c} \simeq \mathbb{Z}/\ell(n-c)[-c]$ by iteration; similarly $L/\ell(n)_{-1} \simeq L/\ell(n-1)[-1]$ and so $L/\ell(n)_{-c} \simeq L/\ell(n-c)[-c]$. Applying Lemma 5, we get canonical isomorphisms

$$H_Z^p(X, \mathbb{Z}/\ell(n)) \simeq H^{p-c}(Z, \mathbb{Z}/\ell(n-c)[-c]) = H^{p-2c}(Z, \mathbb{Z}/\ell(n-c))$$

and

$$H_Z^p(X, L/\ell(n)) \simeq H^{p-c}(Z, L/\ell(n-c)[-c]) = H^{p-2c}(Z, L/\ell(n-c))$$

if $c = \text{codim } Z \leq n$, and both are equal to 0 if $c > n$. The same thing holds in the étale setting, adding subscripts appropriately.

We can now return to our map $\alpha_n : \mathbb{Z}/\ell(n) \rightarrow L/\ell(n)$.

Theorem 6. *Suppose that $BL(n-1)$ holds, and let X be a smooth scheme over k with a subscheme Z of codimension $c > 0$. Then α_n induces isomorphisms on motivic cohomology with supports*

$$H_Z^*(X, \mathbb{Z}/\ell(n)) \xrightarrow{\simeq} H_Z^*(X, L/\ell(n)).$$

Proof. The map in question on cohomology with support, after applying the isomorphisms above, is the map

$$H^{*-2c}(Z, \mathbb{Z}/\ell(n-c)) \rightarrow H^{*-2c}(Z, L/\ell(n-c)),$$

which is just the map on (usual) motivic cohomology induced by α_{n-c} . Since we've assumed $BL(n-1)$, by reverse induction $BL(n-c)$ also holds, and so α_{n-c} is a quasi-isomorphism, so this map is an isomorphism; therefore the map of the theorem is as well. \square

To allow singular X , we extend the definition of $\mathbb{Z}_Z[X]$ in the natural way: for any scheme X with closed subscheme Z , $\mathbb{Z}_{\mathbf{Sm}}[X-Z]$ is a subsheaf of $\mathbb{Z}_{\mathbf{Sm}}[X]$, and we define $\mathbb{Z}_Z[X] = \mathbb{Z}_{\mathbf{Sm}}[X]/\mathbb{Z}_{\mathbf{Sm}}[X-Z]$ and $H_Z^p(X_{\mathbf{Sm}}, \mathcal{F}) = \text{Hom}_{D(\mathbf{Sm}/k)}(\mathbb{Z}_Z[X], \mathcal{F}[p])$ as above. For X smooth, the definitions agree. We get a long exact sequence arising from the defining triangle and a corresponding version of Mayer-Vietoris for closed covers.

We won't prove the theorem for general singular X using this language (indeed I'm not even sure it's true), but we will prove a version for the singular scheme we've had in mind this talk.

Theorem 7. *Let Z be any closed subscheme of $\partial\Delta^m$ which does not contain any of the vertices. If $BL(n-1)$ holds, then*

$$H_Z^*((\partial\Delta^m)_{\mathbf{Sm}}, \mathbb{Z}/\ell(n)) \simeq H_Z^*((\partial\Delta^m)_{\mathbf{Sm}}, L/\ell(n)).$$

Proof. Just as last time we talked about $\partial\Delta^m$, we have spectral sequences converging to $H_Z^{p+q}((\partial\Delta^m)_{\mathbf{Sm}}, \mathcal{F})$ for \mathcal{F} equal to each of $\mathbb{Z}/\ell(n)$ and $L/\ell(n)$, with α_n inducing a map of spectral sequences between them. Each $E_1^{p,q}$ is a product of copies of $H_{Z \cap F}(F, \mathcal{F})$, where F is some intersection of the $\partial_i\Delta^m$. Since Z does not contain any of the vertices, every nonempty $Z \cap F$ has positive codimension in F , and so by Theorem 6 the map induced by α_n is an isomorphism on each term, and therefore is an isomorphism on the E_1 -page. Since this page determines the spectral sequence, the abutments must also be isomorphic, which gives the desired claim. \square

All of this holds modulo higher powers of ℓ , though somewhat more care is needed for the proof of Theorem 6 (in this case the proof as given only works for smooth Z , but we can still get the result via induction on the dimension).

REFERENCES

- [1] Christian Haesemeyer and Charles A. Weibel. *The norm residue theorem in motivic cohomology*. Princeton University Press, 2019.