Norm residue isomorphism theorem: Rost motives

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1. Introduction

Fix $a \in K_n^M(k)$ and suppose we have a Rost variety $X = X_a$ for a. Our goal today is to associate to X a Rost motive $M = (X, e)$ for some idempotent $e : X \to X$ with coefficients in $R = \mathbb{Z}_{(\ell)}$. (We'll always mean $\mathbb{Z}_{(\ell)}$ by R unless stated otherwise.)

Let's first recall what a Rost motive is. Let $\mathfrak{X}$ be the simplicial scheme with objects X^n and face maps given by projections, and as last time write ϵ for $R_{\mathrm{tr}}(\mathfrak{X})$ and ϵM for $\epsilon \otimes M$ for any motive M. We have a structure map $y : M \to \epsilon$ and a twisted dual $y^D : \epsilon\mathbb{L}^d \to M$, where $d = \dim X = \ell^{n-1} - 1$. We say that M is a Rost motive associated to X if

(i) $M = (X, e)$ is a symmetric Chow motive,

(ii) there is a map $\lambda : R_{\mathrm{tr}}(X) \to M$ factoring the projection $R_{\mathrm{tr}}(X) \to R$, and

(iii) there is a motive D fitting into the two distinguished triangles

$$D \otimes \mathbb{L}^b \to M \xrightarrow{y} \epsilon \to, \qquad \epsilon\mathbb{L}^d \xrightarrow{y^D} M \to D \to,$$

where $b = d/(\ell - 1) = 1 + \ell + \cdots + \ell^{n-2}$.

The outline of this lecture is as follows: first, given a class $z \in H^{2b+1,b}(\mathfrak{X}, R) = \mathrm{Hom}(R_{\mathrm{tr}}(X), R(b)[2b+1])$, we will construct a candidate and prove that it satisfies (ii) and (iii). To verify (i), we need to impose a condition on z, namely that it be "suitable"; we'll then show that under this assumption (i) holds, and (using skipped material from section 3) show that such a z exists to conclude that we can use this construction to get a Rost variety for any X.

2. A candidate

Fix $z \in H^{2b+1,b}(\mathfrak{X}, R)$, i.e. a morphism $z : \epsilon \to R(b)[2b+1] = \mathbb{L}^b[1]$ in $\mathbf{DM}^{\mathrm{eff}}_{\mathrm{nis}}(k, R)$. By a result from last time, for any motive N we have $\mathrm{Hom}(\epsilon, N) \cong \mathrm{Hom}(\epsilon, \epsilon N)$ and so z lifts to a morphism $\epsilon \to \epsilon\mathbb{L}^b[1]$, which we will also denote by z. Up to isomorphism, there is a unique A with a map $y : A \to \epsilon$ fitting into the triangle

$$\epsilon\mathbb{L}^b \xrightarrow{x} A \xrightarrow{y} \epsilon \xrightarrow{z} \epsilon\mathbb{L}^b[1] \to .$$

From last time, we have $A \cong A^\dagger(b)[2b] = A^\dagger \otimes \mathbb{L}^b$, and setting $y^D = y^\dagger \otimes \mathbb{L}^b$ taking $\mathfrak{X}$-duals of this triangle and tensoring with $\mathbb{L}^b$ gives the triangle

$$\epsilon\mathbb{L}^b \xrightarrow{y^D} A^\dagger \otimes \mathbb{L}^b \xrightarrow{x^\dagger \otimes \mathbb{L}^b} \epsilon \xrightarrow{z^\dagger[1] \otimes \mathbb{L}^b} \epsilon\mathbb{L}^b \to .$$

Lemma 1. *There is a map $\lambda_1 : R_{\mathrm{tr}}(X) \to A$ factoring the structure map $R_{\mathrm{tr}}(X) \to \epsilon$ as $R_{\mathrm{tr}}(X) \xrightarrow{\lambda} A \xrightarrow{y} \epsilon$.*

Proof. Applying $\mathrm{Hom}(R_{\mathrm{tr}}(X), -)$ to the first triangle above gives the exact sequence

$$\mathrm{Hom}(R_{\mathrm{tr}}(X), A) \xrightarrow{y} \mathrm{Hom}(R_{\mathrm{tr}}(X), \epsilon) \xrightarrow{z} \mathrm{Hom}(R_{\mathrm{tr}}(X), \mathbb{L}^b[1]) = 0$$

by the vanishing theorem ([2, 19.3]). Therefore the map sending $\lambda : R_{\mathrm{tr}}(X) \to A$ to its composition with $y : A \to \epsilon$ is surjective, and in particular there exists some λ_1 whose composition with y is the structure map $R_{\mathrm{tr}}(X) \to \epsilon$ as claimed. $\square$

By composing with the structure map $\epsilon = R_{\mathrm{tr}}(\mathfrak{X}) \to R$, we see that A satisfies (ii), and so we might expect that A is our candidate to be a Rost motive, and indeed if $\ell = 2$ it is: in this case $b = d$ and so the above two triangles show that $D = \epsilon$ satisfies (iii). Showing (i) is harder, but we'll be able to prove it eventually.

In general though this doesn't work: for $\ell > 2$ we no longer have $b = d$, and so the second triangle here does not correspond to the second triangle required for condition (iii). To rectify this, in general we define $M = \mathrm{Sym}^{\ell-1}(A)$ and $D = \mathrm{Sym}^{\ell-2}(A)$, which recovers A and ϵ in the case $\ell = 2$.

It is not immediate that M satisfies either (ii) or (iii). To prove both, we work in the following more general setting. Let $z \in H^{2p+1,q}(\mathfrak{X}, R)$ for arbitrary $p, q \geq 0$, and set $T = R(q)[2p]$, so that z is the same thing as a morphism $\epsilon \to T[1]$ or equivalently $\epsilon \to \epsilon T[1]$. As above, let A be the motive fitting into the triangle

$$\epsilon T \xrightarrow{x} A \xrightarrow{y} \epsilon \xrightarrow{z} \epsilon T[1] \to .$$

We have a slice filtration on A, which we can read off this triangle: $s_0(A) = \epsilon$ and $s_q(A) = \epsilon T$. (This is the filtration $s_{\geq n}(M) = R\,\mathrm{Hom}(\mathbb{L}^n, M) \otimes \mathbb{L}^n$ corresponding to the subcategory generated by $\epsilon R(q)$ with $q \geq n$.)

There is a transfer map $\mathrm{tr} : \mathrm{Sym}^i(A) \to \mathrm{Sym}^{i-1}(A) \otimes A$ sending

$$a_1 \otimes \cdots \otimes a_i \mapsto \sum_j (a_1 \otimes \cdots \widehat{a}_j \otimes \cdots a_i) \otimes a_j$$

and a corestriction map $\mathrm{cores} : S^{i-1}(A) \otimes A \to S^i(A)$ sending $(a_1 \otimes \cdots \otimes a_{i-1}) \otimes a$ to the image of $a_1 \otimes \cdots \otimes a_{i-1} \otimes a$ in $\mathrm{Sym}^i(A) = A^{\otimes i}/S_i$. Let $u = (\mathrm{id} \otimes y) \circ \mathrm{tr} : \mathrm{Sym}^i(A) \to \mathrm{Sym}^{i-1}(A) \otimes A \to \mathrm{Sym}^{i-1}(A) \otimes \epsilon \cong \mathrm{Sym}^{i-1}(A)$ and $v = \mathrm{cores} \circ (\mathrm{id} \otimes x) : \mathrm{Sym}^{i-1}(A) \otimes T \to \mathrm{Sym}^{i-1}(A) \otimes A \to \mathrm{Sym}^i(A)$. We have $\mathrm{Sym}^i T = T^i$ by [2, 15.7], and so $\mathrm{Sym}^i(\epsilon T) \cong \epsilon S^i(T) \cong \epsilon T^i$. Therefore $\mathrm{Sym}^i x$ is a map $\epsilon T^i \to \mathrm{Sym}^i A$.

Proposition 2. *There exist unique morphisms $r : \mathrm{Sym}^{i-1}(A) \to \epsilon T^i[1]$ and $s : \epsilon \to \mathrm{Sym}^{i-1}(A) \otimes T[1]$ such that we have distinguished triangles*

$$\epsilon T^i \xrightarrow{\mathrm{Sym}^i x} \mathrm{Sym}^i(A) \xrightarrow{u} \mathrm{Sym}^{i-1}(A) \xrightarrow{r} \epsilon T^i[1] \to$$

and

$$\mathrm{Sym}^{i-1}(A) \otimes T \xrightarrow{v} \mathrm{Sym}^i(A) \xrightarrow{\mathrm{Sym}^i y} \epsilon \xrightarrow{s} \mathrm{Sym}^{i-1}(A) \otimes T[1],$$

and under the slice filtration on $\mathrm{Sym}^i(A)$, restricted to $s_{<qi}(\mathrm{Sym}^i A)$ u is an isomorphism onto $\mathrm{Sym}^{i-1}(A)$; similarly v identifies $s_{>0}(\mathrm{Sym}^i A)$ with $\mathrm{Sym}^{i-1}(A) \otimes T$; and $\mathrm{Sym}^i y$ identifies ϵ with $s_0(\mathrm{Sym}^i A)$.

Observe that taking $p = q = b$ and $i = \ell - 1$ gives distinguished triangles as in (iii) for $M = \mathrm{Sym}^{\ell-1} A$ and $D = \mathrm{Sym}^{\ell-2} A$, since then $T = \mathbb{L}^b$ and $T^i = \mathbb{L}^{b(\ell-1)} = \mathbb{L}^d$.

Proof. We have $s_n(A^{\otimes i}) = \sum_{n_1 + \cdots + n_i = n} s_{n_1}(A) \otimes \cdots \otimes s_{n_i}(A)$. We can take the symmetric part uniformly to get a decomposition of $s_n(\mathrm{Sym}^i A)$ into terms of the form $s_m(A)$, which are 0 unless $m = 0$ or $m = q$ as above. Thus $s_n(A)$ consists of tensors of up to i copies of things living in degree q as well as potentially sum in degree 0, and therefore $s_n(A)$ is trivial unless $n = qj$ for some $j \le i$.

Since $\epsilon T^i = \epsilon R(q)[2p]$ is concentrated in slice degree qi, $\operatorname{Sym}^i x$ injectively maps it into $s_{qi}(\operatorname{Sym}^i A)$; its cokernel in this part is the image of ϵT^i in $s_{qi}(\operatorname{Sym}^{i-1} A) = 0$ because we cannot have $qi = qj$ for $j \le i-1$. Therefore $\operatorname{Sym}^i x$ identifies ϵT^i with $s_{qi}(\operatorname{Sym}^i A)$ as claimed.

Expanding the definition of u, we see that its part in degree qj, i.e. $s_{qj}(u)$, is just multiplication by $i-j$, which is an isomorphism for $j < i$. The existence and uniqueness of r then follows from the lemma below. Essentially the same argument applies to the second case. □

Lemma 3. *Suppose we have a sequence $A \xrightarrow{a} B \xrightarrow{b} C$ with A in slice degree $\ge n$ and C in slice degree $< n$. If $s_i(a)$ is an isomorphism for $i \ge n$ and $s_i(b)$ is an isomorphism for $i < n$, then there is a unique morphism $c : C \to A[1]$ such that $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ is a distinguished triangle, which identifies A and C with $s_{\ge n}B$ and $s_{<n}B$ respectively.*

Proof. Choose any distinguished triangle $A \xrightarrow{a} B \to C' \to A[1]$ extending a. Since $ba = 0$ (since it is a map from A in degree $\ge n$ to C in degree $< n$), b factors through some $\phi : C' \to C$. For $i < n$, taking s_i of everything $s_i(b)$ is an isomorphism and so $s_i(\phi)$ must be as well; for $i \ge n$, $s_i(a)$ is an isomorphism and so $s_i(B \to C')$ must be the zero map, as is $s_i(B \to C)$ since C is in degree $< n$, and so $s_i(\phi)$ is trivially an isomorphism. Therefore $s_i(\phi)$ is always an isomorphism and so ϕ must be as well. Thus precomposing $C' \to A[1]$ with ϕ^{-1} gives the desired triangle.

To get uniqueness, suppose that (a, b, c') is a second triangle on (A, B, C). Then there is an endomorphism $f : C \to C$ giving a morphism of triangles $(\mathrm{id}_A, \mathrm{id}_B, f)$. Since $s_{<n}(b)$ is an isomorphism, $s_{<n}(f)$ must be the identity.

Finally, the last assertion follows since $s_{\ge n}C = 0$ and so $s_{\ge n}(a) : s_{\ge n}A = A \to s_{\le n}B$ is an isomorphism, and similarly for $s_{<n}(b) : s_{<n}B \to s_{<n}C = C$. □

Next, we want to show that M also satisfies (ii).

Proposition 4. *There exists a map $\lambda : R_{\mathrm{tr}}(X) \to M = \operatorname{Sym}^{\ell-1}(A)$ factoring $R_{\mathrm{tr}}(X) \to \epsilon$ as $R_{\mathrm{tr}}(X) \xrightarrow{\lambda} \operatorname{Sym}^{\ell-1}(A) \xrightarrow{\operatorname{Sym}^{\ell-1} y} \epsilon$.*

Proof. Recall that $R_{\mathrm{tr}}(X) \to \epsilon$ factors through $\lambda_1 : R_{\mathrm{tr}}(X) \to A$ by Lemma 1. Taking this as the base case, we'll show that for every $i < \ell$ we have a map $\lambda_i : R_{\mathrm{tr}}(X) \to \operatorname{Sym}^i(A)$ factoring $R_{\mathrm{tr}}(X) \to \epsilon$.

Applying $\operatorname{Hom}(R_{\mathrm{tr}}(X), -)$ to the first triangle of Proposition 2 gives the exact sequence

$$\operatorname{Hom}(R_{\mathrm{tr}}(X), \operatorname{Sym}^i(A)) \xrightarrow{u} \operatorname{Hom}(R_{\mathrm{tr}}(X), \operatorname{Sym}^{i-1}(A)) \xrightarrow{r} \operatorname{Hom}(R_{\mathrm{tr}}(X), \epsilon T^i[1]) = 0$$

by the vanishing theorem again, so in particular $\lambda_{i-1} : R_{\mathrm{tr}}(X) \to \operatorname{Sym}^{i-1}(A)$ factors as $\lambda_{i-1} = u \circ \lambda_i$ for some $\lambda_i : R_{\mathrm{tr}}(X) \to \operatorname{Sym}^i(A)$. By induction we conclude that $R_{\mathrm{tr}}(X) \to \epsilon$ factors as $y \circ u \circ \cdots \circ u \circ \lambda_{\ell-1}$ since $u \circ \cdots \circ u \circ \lambda_{\ell-1} = \lambda_1$ and we know from Lemma 1 that $y \circ \lambda_1$ is the desired structure map. But $yu^{\ell-1} = \operatorname{Sym}^{\ell-1} y$ by the definition of u, and so the result follows. □

3. Suitable cohomology classes

It remains to show that our candidate $M = \mathrm{Sym}^{\ell-1}(A)$ satisfies (i), i.e. is a symmetric Chow motive, where A is constructed as above from a cohomology class $z \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}_{(\ell)})$. In fact this is not true for arbitrary z; but we will impose a condition on it that makes it true.

We need the motivic cohomology operations $Q_i : H^{*,*}(X, \mathbb{Z}_\ell) \to H^{*+2\ell^i-1,\ell^i-1}(X, \mathbb{Z}/\ell)$. These are defined by $Q_0 = \beta$ and $Q_{i+1} = P^{\ell^i}Q_i - Q_i P^{\ell^i}$ (for $\ell > 2$), where P^i are Voevodsky's reduced power operations and β is the Bockstein map.

Definition. We say that $z \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}_{(\ell)})$ is suitable if its mod ℓ reduction $\bar{z}$ satisfies $Q_{n-1}(\bar{z}) \neq 0$ and $Q_i(\bar{z}) = 0$ for every $0 \leq i < n-1$.

If $z = \tilde{\beta}Q_1 \cdots Q_{n-2}(\delta)$, where $\tilde{\beta}$ is the integral Bockstein, then z is suitable if and only if $Q_{n-1}(\bar{z}) \neq 0$, since $\bar{z} = Q_0 \cdots Q_{n-2}\delta$ and so $Q_i(\bar{z}) = 0$ since the Q_i anticommute and square to 0.

From the part of section 3 that we skipped, since X splits $a \neq 0$, assuming $\mathrm{BL}(n-1)$ there is a unique nonzero lift $\delta \in H^{n,n-1}(\mathfrak{X}, \mathbb{Z}/\ell) = H^n(\mathfrak{X}, \mathbb{Z}/\ell(n-1))$ of $a \in K_n^M(k) = H^n(X, \mathbb{Z}/\ell(n))$, and $\mu = \tilde{\beta}Q_1 \cdots Q_{n-2}\delta \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}/\ell)$ satisfies $Q_{n-1}(\mu) \neq 0$ and therefore is suitable.

We now need the notion of the fundamental class τ of X. Recall that by motivic duality $R_{\mathrm{tr}}(X) \simeq R_{\mathrm{tr}}(X)^* \otimes \mathbb{L}^d$, and so

$$\mathrm{Hom}(\mathbb{L}^d, R_{\mathrm{tr}}(X)) \cong \mathrm{Hom}(\mathbb{L}^d, R_{\mathrm{tr}}(X)^* \otimes \mathbb{L}^d) \cong \mathrm{Hom}(R_{\mathrm{tr}}(X), R) = H^0(X, R).$$

The cohomology $H^*(X, R)$ is a ring and in particular has an identity $1 \in H^0(X, R)$, which thus corresponds to a canonical map $\tau : \mathbb{L}^d \to R_{\mathrm{tr}}(X)$ and thus another map $\epsilon\mathbb{L}^d \to R_{\mathrm{tr}}(X)$, also denoted τ.

Proposition 5. *Let $z \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}/\ell)$ be such that $Q_{n-1}(z) \neq 0$ and $Q_i(z) = 0$ for $0 \leq i < n-1$. Then the composition*

$$\epsilon\mathbb{L}^d \xrightarrow{\tau} R_{\mathrm{tr}}(X) \xrightarrow{\lambda} S^{\ell-1}(A)$$

is nonzero, where $R = \mathbb{Z}/\ell$.

This follows from more magic in section 13 and our factorization of the structure map.

Since symmetric powers commute with duals and $A \cong A^\dagger \otimes \mathbb{L}^b$, we have $\mathrm{Sym}^i(A) \cong \mathrm{Sym}^i(A)^\dagger \otimes \mathbb{L}^{bi}$. Therefore we can define the dual map

$$\lambda^D : \mathrm{Sym}^{\ell-1}(A) \cong \mathrm{Sym}^{\ell-1}(A)^\dagger \otimes \mathbb{L}^d \xrightarrow{\lambda^\dagger \otimes \mathbb{L}^d} R_{\mathrm{tr}}(X)^\dagger \otimes \mathbb{L}^d \cong R_{\mathrm{tr}}(X).$$

Theorem 6. *Suppose that $z \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}_{(\ell)})$ is suitable. Then the composition $\lambda \circ \lambda^D$ is an isomorphism on $M = \mathrm{Sym}^{\ell-1}(A)$, and there is a constant $c \in \mathbb{Z}_{(\ell)}$ such that $\lambda \circ \tau = c\,\mathrm{Sym}^{\ell-1} x$ and the following diagram commutes:*

$$\begin{array}{ccc} M & \xrightarrow{\lambda\circ\lambda^D} & M \\ \big\downarrow{\scriptstyle \mathrm{Sym}^{\ell-1} y} & & \big\downarrow{\scriptstyle \mathrm{Sym}^{\ell-1} y} \\ \epsilon & \xrightarrow{c} & \epsilon \end{array}.$$

In particular λ^D splits λ, so that $M = \mathrm{Sym}^{\ell-1}(A)$ is a direct summand of $R_{\mathrm{tr}}(X)$ and so a Chow motive.

Proof. By Proposition 2, $\mathrm{Sym}^{\ell-1} y : M = \mathrm{Sym}^{\ell-1} A \to \epsilon$ is the projection onto $s_0(M)$. From last time $\mathrm{End}(\epsilon) = R$, so restricting to s_0 the diagram commutes for some c given by $s_0(\lambda \circ \lambda^D)$. By a result we skipped, $\mathrm{End}(M)$ is a local ring with $s_0 : \mathrm{End}(M) \twoheadrightarrow R$ a local homomorphism, so $c = s_0(\lambda \circ \lambda^D)$ is a unit if and only if $\lambda \circ \lambda^D$ is an isomorphism, in which case $e = \lambda^D \circ (\lambda \circ \lambda^D)^{-1} \circ \lambda$ is idempotent and $M = eR_{\mathrm{tr}}(X)$ is a direct summand of $R_{\mathrm{tr}}(X)$, so we're done. Therefore it suffices to prove that there exists $c \not\equiv 0 \pmod{\ell}$ making the diagram commute.

From last time, $\mathrm{Hom}(\epsilon\mathbb{L}^r, \epsilon) = 0$ for $r > 0$. Applying $\mathrm{Hom}(\epsilon\mathbb{L}^r, -)$ to the second triangle from Proposition 2 gives an exact sequence

$$\mathrm{Hom}(\epsilon\mathbb{L}^r, \mathrm{Sym}^{i-1} A \otimes \mathbb{L}^b) \to \mathrm{Hom}(\epsilon\mathbb{L}^r, \mathrm{Sym}^i A) \to \mathrm{Hom}(\epsilon\mathbb{L}^r, \epsilon) = 0,$$

and so by induction on i we have $\mathrm{Hom}(\epsilon\mathbb{L}^r, \mathrm{Sym}^i A) = 0$ for $r > bi$. Applying $\mathrm{Hom}(\epsilon\mathbb{L}^d, -)$ to the first triangle from the same proposition with $i = \ell - 1$ gives an exact sequence

$$\mathrm{Hom}(\epsilon\mathbb{L}^d, \epsilon\mathbb{L}^d) \xrightarrow{\mathrm{Sym}^{\ell-1} x} \mathrm{Hom}(\epsilon\mathbb{L}^d, \mathrm{Sym}^{\ell-1} A) \to \mathrm{Hom}(\epsilon\mathbb{L}^d, \mathrm{Sym}^{\ell-2} A).$$

Recalling that A lives in the subcategory in degree $\leq b$ for the slice filtration, and in particular $\mathrm{Sym}^{\ell-2} A \otimes \mathbb{L}^{-d}$ lives in degree $\leq (\ell-2)b - d = (\ell-2)b - (\ell-1)b = -b < 0$ and so $H^0(X, \mathrm{Sym}^{\ell-2} A \otimes \mathbb{L}^{-d}) = \mathrm{Hom}(\epsilon, \mathrm{Sym}^{\ell-2} A \otimes \mathbb{L}^{-d}) = \mathrm{Hom}(\epsilon\mathbb{L}^d, \mathrm{Sym}^{\ell-2} A) = 0$. Therefore every map $\epsilon\mathbb{L}^d \to M$ lifts to an endomorphism of $\epsilon\mathbb{L}^d$, i.e. an endomorphism of ϵ, which is just a constant; in particular there exists some constant C lifting the composite $\lambda \circ \tau : \epsilon\mathbb{L}^d \to R_{\mathrm{tr}}(X) \to M$, i.e. $\lambda \circ \tau = C \,\mathrm{Sym}^{\ell-1} x$. Since this composite is nonzero modulo ℓ by Proposition 5, we conclude that $C \neq 0 \pmod{\ell}$.

Dualizing gives the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "\lambda^D"] \arrow[d, "\mathrm{Sym}^{\ell-1} y"'] & R_{\mathrm{tr}}(X) \arrow[r, "\lambda"] \arrow[d] & M \arrow[dl, "\mathrm{Sym}^{\ell-1} y"] \\
\epsilon \arrow[r, "C"] & \epsilon &
\end{tikzcd}$$

since $\mathrm{Sym}^{\ell-1} y$ is dual to $\mathrm{Sym}^{\ell-1} x$ (essentially by Proposition 2) and the structure map $R_{\mathrm{tr}}(X) \to \epsilon$ is dual to τ, with the right triangle commuting by Proposition 4. Since $\mathrm{Sym}^{\ell-1} y$ is the s_0-projection, it follows that $C = s_0(\lambda \circ \lambda^D)$ as desired, and so since we know $C \neq 0 \pmod{\ell}$ the claim follows. $\square$

Corollary 7. *For $z \in H^{2b+1,b}(\mathfrak{X}, \mathbb{Z}_{(\ell)})$ suitable, the corresponding motive M is a symmetric Chow motive.*

Proof. Choose $e = c^{-1}\lambda^D \circ \lambda : R_{\mathrm{tr}}(X) \to R_{\mathrm{tr}}(X)$. By Theorem 6 $M = \mathrm{Sym}^{\ell-1}(A) \cong (X, e)$ is a Chow motive, with transpose (X, e^t) defined by

$$\lambda^t : M \cong M^* \otimes \mathbb{L}^d \xrightarrow{\lambda^* \otimes \mathbb{L}^d} R_{\mathrm{tr}}(X)^* \otimes \mathbb{L}^d \cong R_{\mathrm{tr}}(X).$$

We have $\mathrm{Hom}(M, \epsilon) \cong \mathrm{Hom}(M, R_{\mathrm{tr}}(X))$, which identifies λ^t with λ^D; therefore $e = c^{-1}\lambda^t \circ \lambda$ satisfies $e^t = e$, i.e. $M = (X, e)$ is a symmetric Chow motive. $\square$

Corollary 8. *For every $a \in K_n^M(k)$ and Rost variety X for a, there exists a Rost motive M for a.*

Proof. We know that there exists a suitable z, namely μ from section 3; by Propositions 2 and 4 we know that the corresponding motive $M = \mathrm{Sym}^{\ell-1}(A)$ satisfies conditions (ii) and (iii), and by Corollary 7 it also satisfies (i) and therefore is a Rost variety for a. □

References

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