Geometrization of real local Langlands: speculation

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The goal of this note is to write down as much as I can understand from Scholze's Noether lectures on geometrization of real local Langlands (primarily the third and final lecture), and to speculate on some of his remarks. Obviously no material is original, except for any mistakes, which will likely be manifold.

For convenience I am going to pretend that I completely understand the various condensed/liquid/gaseous/analytic structures on the various real and complex spaces and groups; Scholze spent some time clarifying which is used where, but I will mostly avoid the issue and trust that everything works out. The reader is encouraged to think of these as topological structure which is miraculously homologically well-behaved.

1. Big picture: desiderata

Let's first briefly review how geometrization of p-adic local Langlands à la Fargues–Scholze works. Classical local Langlands seeks to parametrize smooth representations of $G(F)$ in terms of L-parameters for $\check{G}$, where G is a reductive group with Langlands dual group $\check{G}$ and F is a local field; L-parameters for $\check{G}$ consist roughly of Galois representations $\mathrm{Gal}(\overline{F}/F) \to \check{G}$ together with some additional data. This parametrization should satisfy an assortment of properties. The idea of Fargues–Scholze is that when F is nonarchimedean all of this data should somehow come from geometric objects, on the level of which we can restate the local Langlands conjecture categorically. In particular, we look at the (suitably defined) derived category of smooth representations $D([*/G(F)], \overline{\mathbb{Q}}_\ell)$ and find some stack of L-parameters $Z_{F,\check{G}}$; then the parametrization can be upgraded to a fully faithful functor $D([*/G(F)], \overline{\mathbb{Q}}_\ell) \to D_{\mathrm{qc}}(Z_{F,\check{G}})$, which again should satisfy certain properties.[1]

We might hope to somehow upgrade this functor to an equivalence. This can be done by geometrizing the automorphic side as well as the Galois side: there is a stack Bun_G such that $[*/G(F)]$ embeds into Bun_G, and pushforward along this embedding gives an embedding $D([*/G(F)]) \to D(\mathrm{Bun}_G)$. (There are also other strata in Bun_G corresponding to inner twists of G.) This larger category should then be conjecturally equivalent to the Galois side $D_{\mathrm{qc}}(Z_{F,\check{G}})$, again after imposing suitable compatibilities and caveats.

In turn, we can define Bun_G geometrically via the Fargues–Fontaine curve: for every object S in our test category (perfectoid spaces of characteristic p), we can form a "curve" X_S which replaces the product $X \times S$ for a base curve X as would appear in geometric Langlands. Then Bun_G is the stack sending S to the space of G-bundles on X_S.

Our goal is to carry out a similar program for $F = \mathbb{R}$. We proceed in reverse order: first we want to find a suitable replacement X_S for the Fargues–Fontaine curve over the reals; this also entails finding a good replacement for our test category, which we can think of as the analogue of (characteristic p) perfectoid spaces. We can then form Bun_G extending

[1] Of course, one has to define these categories more carefully to make sense of this; there are also various caveats, e.g. following the geometric Langlands program we should really enforce some "nilpotence" condition on this derived category on the right.

$[*/G(\mathbb{R})]$, and then look for a geometrization of L-parameters. Finally we'll speculate about some examples and related ideas.

It's worth observing here that already at the level of $[*/G(\mathbb{R})]$, there is quite a lot of technical difficulty. First, what does $*$ mean? Since we're over $\mathbb{R}$, presumably it should mean $\operatorname{Spec}\mathbb{R}$; but we should take the topology of $\mathbb{R}$ into account. In the p-adic setting, we can use adic spaces to write $\operatorname{Spa}\mathbb{Q}_p$ instead of $\operatorname{Spec}\mathbb{Q}_p$ for this sort of thing; but for archimedean fields this no longer works. Instead, we should use the machinery of analytic stacks: equip $\mathbb{R}$ with a suitable analytic ring structure, coming from a liquid or gaseous structure, and take $* = \operatorname{AnSpec}(\mathbb{R})$. The same concern applies to $G(\mathbb{R})$: this is a topological group with $\mathbb{R}$-structure and so we think of it as a group in analytic stacks over $\mathbb{R}$, equipping it with the corresponding analytic structure. Other topological $\mathbb{R}$-groups that may arise should generally be thought of similarly.

2. Twistor $\mathbb{P}^1$

The first order of business is to find a replacement for the Fargues–Fontaine curve at the infinite place. For somewhat mysterious reasons, the right object turns out to be "twistor $\mathbb{P}^1_{\mathbb{R}}$, which we write as $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ to avoid confusion.

Ultimately, we'll want to have a test category $\mathcal{C}$ and a relative construction $\widetilde{\mathbb{P}}^1_{\mathbb{R},A}$, which analogously to the Fargues–Fontaine curve is not given in general by the base change but by some more complicated construction. First, though, let's just describe the usual $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ to try to get some intuition.

One definition is: $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ is the unique nonsplit real form of $\mathbb{P}^1_{\mathbb{C}}$. What this means is: $\mathbb{P}^1_{\mathbb{C}}$ can be thought of as the Riemann sphere, and is equipped with an antiholomorphic involution $\rho(z) = \bar{z}$, which we can think of as defining a real form: indeed the fixed points of this involution are exactly the real points. We can think of ρ as a descent datum to $\mathbb{R}$, which gives the scheme $\mathbb{P}^1_{\mathbb{R}}$ over $\mathbb{R}$ whose base change to $\mathbb{C}$ recovers $\mathbb{P}^1_{\mathbb{C}}$.

One could instead take the involution $\rho(z) = -1/\bar{z}$. This is again a Galois descent datum and so we can find a scheme $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ over $\mathbb{R}$ whose base change to $\mathbb{C}$ is $\mathbb{P}^1_{\mathbb{C}}$, but it is very different-looking: in particular it has no real points, since ρ has no fixed points. This is the unique such real form of $\mathbb{P}^1_{\mathbb{C}}$. We could define it more directly for example as the projective variety cut out by $x^2 + y^2 + z^2 = 0$, which is a genus 0 curve defined over $\mathbb{R}$ but has no real points.

Why might this be of interest for the archimedean local Langlands program? Recall that hermitian symmetric domains (the "archimedean part" of Shimura varieties) are moduli spaces of Hodge structures. And in fact $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ is closely related to Hodge structures!

Explicitly, fix a point at infinity $\infty : \operatorname{Spec}\mathbb{C} \to \widetilde{\mathbb{P}}^1_{\mathbb{R}}$. Away from ∞, the automorphism group G of $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ over $\operatorname{Spec}\mathbb{R}$ is a non-split form of $\mathbb{G}_{\mathrm{m}}$; pulling back along the structure map identifies $\mathbb{R}$-vector spaces with G-equivariant vector bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}} - \{\infty\}$. The data of extending a G-equivariant semistable vector bundle of slope s on $\widetilde{\mathbb{P}}^1_{\mathbb{R}} - \{\infty\}$ to the point at infinity is precisely the data of a pure Hodge structure of weight $2s$ on the corresponding $\mathbb{R}$-vector space: that is, pullback along the structure map induces an equivalence of categories between pure Hodge structures of weight n and G-equivariant semistable vector bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ of slope $\frac{n}{2}$. Dropping the equivariance condition gives a larger category of *twistor structures* in which Hodge structures are the G-equivariant objects, and which are equivalent to semistable vector bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ of given slope.

Now, we need a relative version of this construction to define Bun_G, and base change to our test objects does not work. Scholze sketched some geometric desiderata to discuss what the relative $\widetilde{\mathbb{P}}^1_{\mathbb{R},A}$ should be, but I did not follow them very well; instead, I'll try to sketch the explicit construction for certain "nil-perfectoid" test objects A. Here the "perfectoid" rings are rings of continuous functions $\mathrm{Cont}(S, \mathbb{C})$ for suitable Hausdorff spaces S; presumably this should be interpreted in a reasonably condensed way. To more general test rings A we associate an ideal $\mathrm{Nil}^\dagger(A)$, which is inspired by a p-adic construction of Rodriguez-Camargo; the nil-perfectoid rings are those A such that $A/\mathrm{Nil}^\dagger(A) \simeq \mathrm{Cont}(S, \mathbb{C})$ for some S as above.

For such A, we can proceed as follows. The usual projective line $\mathbb{P}^1_{\mathbb{C}}$ can be covered by two copies of $\mathbb{A}^1_{\mathbb{C}}$, one away from ∞ and one away from 0. Here, our involution ρ exchanges 0 and ∞, so we can actually cover the descended $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ by a single copy of $\mathbb{A}^1_{\mathbb{C}}$; note that this should be viewed as an analytic space, so we write this cover as $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \to \widetilde{\mathbb{P}}^1_{\mathbb{R}}$. In particular we can view $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ as the quotient of $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ by some equivalence relation. Now for any $\mathbb{C}$-algebra A as above, we can define $\mathbb{A}^1_A = \mathrm{AnSpec}\, A[\lambda]$ and form the pullback $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}} \times_{(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{alg}}} \mathrm{AnSpec}\, A[\lambda]/(\lambda \,\mathrm{Nil}^\dagger(A)) \to (\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$; here λ is the coordinate on $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ rather than an independent variable. Quotienting by the pullback of the equivalence relation on $(\mathbb{A}^1_{\mathbb{C}})^{\mathrm{an}}$ gives the relative twistor line $\widetilde{\mathbb{P}}^1_{\mathbb{R},A}$. (A priori this is only defined over $\mathbb{C}$; but restricted to $\mathbb{G}_{\mathrm{m}}^{\mathrm{an}}$, i.e. for λ invertible, we are restricted to $\mathrm{AnSpec}\, A/\mathrm{Nil}^\dagger(A) = \mathrm{AnSpec}\,\mathrm{Cont}(S, \mathbb{C})$ since A is nil-perfectoid, and $\mathrm{Cont}(S, \mathbb{C})$ descends canonically to the real algebra $\mathrm{Cont}(S, \mathbb{R})$.)

We can now define Bun_G straightforwardly to be the stack sending A to G-bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R},A}$. For example one can compute that

$$\mathrm{Bun}_{\mathbb{G}_{\mathrm{m}}} = \bigsqcup_{n \in \mathbb{Z}} [*/\mathbb{R}^\times],$$

where $* = \mathrm{AnSpec}\,\mathbb{R}$ and $\mathbb{R}^\times$ as an $\mathbb{R}$-group should be interpreted as above; we can think of this as one term for every degree n, and so we say that a line bundle $\mathcal{L}$ on $\widetilde{\mathbb{P}}^1_{\mathbb{R},A}$ has degree n if it factors through the nth component.

We can also now compare some properties of the twistor $\mathbb{P}^1$ to the Fargues–Fontaine curve. In particular, vector bundles on the Fargues–Fontaine curve are classified by isocrystals; more generally, G-bundles are classified by Kottwitz's set $B(G, F)$, i.e. $|\,\mathrm{Bun}_G\,| \simeq B(G, F)$. Kottwitz also defined $B(G, F)$ for F real, so one might hope that the same thing is true for our definition; and indeed it is, i.e. G-bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ are classified by $B(G, \mathbb{R})$ and so $|\,\mathrm{Bun}_G\,| \simeq B(G, \mathbb{R})$.

In particular, for p-adic fields the vector bundles on the Fargues–Fontaine curve decompose into direct sums of bundles $\mathcal{O}(\lambda)$ for all rational numbers λ; the rank of $\mathcal{O}(\lambda)$ is the denominator of λ. For the twistor $\mathbb{P}^1$, a similar property holds, but instead of any rational λ we restrict to half-integers $\lambda \in \frac{1}{2}\mathbb{Z}$; this has to do with the fact that $\mathbb{Q}_p$ has extensions of any degree, but $\mathbb{R}$ only has extensions of degree 1 or 2.

3. L-parameters

Just as in the p-adic or geometric cases, in order to discuss the correspondence we first need a notion of Hecke operators; and to define Hecke operators we need a stack of degree 1 divisors on our curve. In the geometric setting, degree 1 divisors on X are precisely parametrized by X; but in the p-adic setting we now have a different stack Div^1 whose S-points are degree 1 divisors of X_S. This turns out to be equivalent to the data of an *untilt* of S.

We proceed by analogy in the real case: let Div^1 be the moduli space of degree 1 divisors on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$, i.e. $\mathrm{Div}^1(A)$ consists of pairs $(\mathcal{L}, s)$ where $\mathcal{L}$ is a degree 1 line bundle on $\widetilde{\mathbb{P}}^1_{\mathbb{R},A}$ in the sense described above and $s \in H^0(\mathcal{L})$ is a nonzero section.

There is a natural 2-to-1 cover $\mathbb{P}^1_{\mathbb{C}\times A} \to \widetilde{\mathbb{P}}^1_{\mathbb{R},A}$, so we can equivalently think of a degree 1 divisor on the base as a degree 1 divisor on $\mathbb{P}^1_{\mathbb{C}\times A}$ together with a descent datum. A divisor on $\mathbb{P}^1_{\mathbb{C}\times A}$ should be a nonzero point of $(\mathbb{A}^2_{\mathbb{C}\times A})^{\mathrm{an}}$ up to $\mathbb{C}^\times$-action; the $\mathbb{C}^\times$-action together with the Galois action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ gives an action of a nonsplit extension $W_{\mathbb{R}}$ of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times$, the Weil group of $\mathbb{R}$ (interpreted as usual as an analytic group), such that (by varying A)

$$\mathrm{Div}^1 = [(\mathbb{A}^2_{\mathbb{C}})^{\mathrm{an}} - \{0\}]/W_{\mathbb{R}}.$$

Now, vector bundles on $\mathrm{Div}^1 = [(\mathbb{A}^2_{\mathbb{C}})^{\mathrm{an}} - \{0\}]/W_{\mathbb{R}}$ extend over the puncture to $(\mathbb{A}^2_{\mathbb{C}})^{\mathrm{an}}/W_{\mathbb{R}}$, and here are in turn closely related to $W_{\mathbb{R}}$-representations, via pulling back along the structure map to $[*/W_{\mathbb{R}}]$ and the zero section $[*/W_{\mathbb{R}}] \to (\mathbb{A}^2_{\mathbb{C}})^{\mathrm{an}}/W_{\mathbb{R}}$; apparently these are likely equivalent but this needs more work. In any case they are very close and it makes sense to define our moduli space of L-parameters to be $\mathrm{Bun}_{\check{G}}(\mathrm{Div}^1)$, the stack over $\mathbb{C}$ sending A to $\check{G}$-bundles on $\mathrm{Div}^1 \times_{\mathbb{C}} \mathrm{AnSpec}\, A$.

With Div^1 defined, we can write down the Hecke diagram:

$$\mathrm{Bun}_G \longleftarrow \mathrm{Hk} \longrightarrow \mathrm{Bun}_G \times \mathrm{Div}^1$$

where Hk is the stack sending A to tuples $(\mathcal{E}, \mathcal{E}', D, \alpha)$ where $\mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_G(A)$, $D \in \mathrm{Div}^1(A)$, and α is an isomorphism $\mathcal{E}|_{\widetilde{\mathbb{P}}^1_{\mathbb{R},A} - D} \simeq \mathcal{E}'|_{\widetilde{\mathbb{P}}^1_{\mathbb{R},A}}$.

4. Variations of twistor structure: complex diamondization

Now that we have a notion of Div^1, recall that in the p-adic situation on the level of diamonds it turns out that Div^1 is (the quotient by Frobenius of) $\mathrm{Spd}\, \breve{\mathbb{Q}}_p$, so for perfectoid spaces over $\mathrm{Spa}\, \overline{\mathbb{F}_p}$ it classifies their characteristic 0 untilts (up to Frobenius). Untilts more generally allow one to define a "diamondization" functor $X \mapsto X^\diamondsuit$, where $X^\diamondsuit$ is the v-sheaf sending a characteristic p perfectoid space S to the set of pairs $(S^\sharp, f)$ where $S^\sharp$ is an untilt of S and f is a morphism $S^\sharp \to X$; in other words, $X^\diamondsuit$ classifies "untilts over X."

In the archimedean setting, even with our analogue of perfectoid spaces it's hard to naively see what tilting should mean: there is no characteristic p to tilt to! However, one desideratum is that, analogous to the p-adic case, *untilts* should be classified by Div^1: and so we can simply replace our notion of untilts with points of Div^1. Thus the complex analogue of diamondization should be the analytic stack $X^\diamondsuit$ sending A to $\{(D, f)\}$ where $D \subset \widetilde{\mathbb{P}}^1_{\mathbb{R},A}$ is a degree 1 divisor, i.e. an A-point of Div^1, and $f : D \to X$ is a morphism. In particular we get a map $X^\diamondsuit \to \mathrm{Div}^1$ for every X.

This should be thought of as related to an archimedean version of analytic prismatization (which in fact motivates a lot of these constructions): in particular one can consider the fibers of this map $X^\diamondsuit \to \mathrm{Div}^1$. Generically, these look like the analytic de Rham space $X^{\mathrm{an}}_{\mathrm{dR}}$, while over the "special fiber" $D_\infty \in \mathrm{Div}^1$ (corresponding to the point at infinity $\infty \in \widetilde{\mathbb{P}}^1_{\mathbb{R}}$) the fiber looks like the analytic Hodge–Tate stack $X^{\mathrm{an}}_{\mathrm{HT}}$. Expanding definitions, we expect that for A (real) "perfectoid" (so in particular $\mathrm{Nil}^\dagger(A) = 0$) we have $\mathrm{Div}^1(A)$ essentially just points of A and so $(\mathrm{AnSpec}\, A)^\diamondsuit \simeq \mathrm{AnSpec}\, A$, as for characteristic p perfectoid spaces.

A different perspective is that vector bundles on $X^\diamondsuit$ are "variations of twistor structures": we have natural covering maps $(\mathbb{A}^2_{\mathbb{C}})^{\mathrm{an}} - \{0\} \to \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbb{P}}^1_{\mathbb{R}}$, inducing $X^\diamondsuit \to \mathrm{Div}^1 \to \widetilde{\mathbb{P}}^1_{\mathbb{R}}/W_{\mathbb{R}}$. In particular vector bundles on $X^\diamondsuit$ can be viewed as deformations of pullbacks of $W_{\mathbb{R}}$-equivariant vector bundles on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$, which is to say variations of twistor structures.

5. Example: isomorphism of Lubin–Tate and Drinfeld towers

Say $G = \mathrm{GL}_2$. In the p-adic case, we can consider the two rank 2 vector bundles $\mathcal{O}^{\oplus 2}$ and $\mathcal{O}(1/2)$ on the Fargues–Fontaine curve; the piece of the Hecke stack where we require the modification to be an injection $\mathcal{O}^{\oplus 2} \hookrightarrow \mathcal{O}(1/2)$, supported at some $D \in \mathrm{Div}^1$, gives $\{\mathcal{O}^{\oplus 2} \hookrightarrow \mathcal{O}(1/2)\} \to \mathrm{Div}^1$. This can be identified with the Lubin–Tate or Drinfeld tower; we have de Rham and Hodge–Tate period maps to $(\mathbb{P}^1)^\diamondsuit$ and Drinfeld's p-adic upper half-plane $\Omega^\diamondsuit$ over Div^1, which are torsors for $\mathrm{GL}_2(\mathbb{Q}_p)$ and a twist $G_b(\mathbb{Q}_p)$ (in this case the p-adic quaternions) respectively.

Both of these vector bundles still make sense on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$, so we would like to do something similar at the infinite place; and now that we have a suitable notion of diamondization, we can. Drinfeld's upper half-space is replaced by the usual upper (or lower) half-space $\mathcal{H}^\pm$, and so we get a commutative diagram

$$\begin{array}{ccc} \{\mathcal{O}^{\oplus 2} \hookrightarrow \mathcal{O}(1/2)\} & \longrightarrow & (\mathbb{P}^1_{\mathbb{C}})^\diamondsuit \\ \downarrow & & \downarrow \\ (\mathcal{H}^\pm)^\diamondsuit & \longrightarrow & \mathrm{Div}^1 \end{array} .$$

The top map is a $\mathrm{GL}_2(\mathbb{R})$-torsor while the left map is a $\mathbb{H}^\times$-torsor, where $\mathbb{H}$ are the (usual) quaternions and again both groups are understood analytically.

Now, $(\mathbb{P}^1_{\mathbb{C}})^\diamondsuit$ can be understood as the quotient of $\{\mathcal{O}^{\oplus 2} \hookrightarrow \mathcal{O}(1/2)\}$ by the $\mathrm{GL}_2(\mathbb{R})$-action, so it has a natural map to $[*/\mathrm{GL}_2(\mathbb{R})]$; the $\mathbb{H}^\times$-action on the space of modifications induces a residual one on $(\mathbb{P}^1_{\mathbb{C}})^\diamondsuit$ which does not affect this structure map and so we get a morphism $[(\mathbb{P}^1_{\mathbb{C}})^\diamondsuit/\mathbb{H}^\times] \to [*/\mathrm{GL}_2(\mathbb{R})]$. On the other hand, the structure morphism $(\mathbb{P}^1_{\mathbb{C}})^\diamondsuit \to \mathrm{Div}^1$ induces $[(\mathbb{P}^1_{\mathbb{C}})^\diamondsuit/\mathbb{H}^\times] \to \mathrm{Div}^1 \times [*/\mathbb{H}^\times]$, so we get a correspondence

$$\begin{array}{ccccc} & & [(\mathbb{P}^1_{\mathbb{C}})^\diamondsuit/\mathbb{H}^\times] & & \\ & \swarrow^{g} & & \searrow^{f} & \\ \mathrm{Div}^1 \times [*/\mathbb{H}^\times] & & & & [*/\mathrm{GL}_2(\mathbb{R})] \end{array} .$$

Sheaves on the bottom right are representations of $\mathrm{GL}_2(\mathbb{R})$, while sheaves on the bottom left should be (generated by) tensor products of sheaves on Div^1, which are morally $W_{\mathbb{R}}$-representations, and sheaves on $[*/\mathbb{H}^\times]$, i.e. $\mathbb{H}^\times$-representations. There are recipes from going from $\mathrm{GL}_2(\mathbb{R})$-representations π to each of these: for the first, associate to π its L-parameter ρ_π via the local Langlands correspondence, and for the second associate to π its Jacquet–Langlands transfer $\mathrm{JL}(\pi)$. On the other hand, the above diagram gives a geometric recipe, namely $\pi \mapsto g_* f^* \pi$, which we can think of as sending π to a $\mathbb{H}^\times$-equivariant variation of twistor structures on $\mathbb{P}^1_{\mathbb{C}}$ and then taking its cohomology. Scholze has proven that when π is a discrete series, these constructions agree:

$$g_* f^* \pi \simeq \rho_\pi \otimes \mathrm{JL}(\pi).$$

Using the other side of the diagram, we can similarly get a function $(\mathcal{H}^\pm)^\diamondsuit \to [*/\mathbb{H}^\times]$ which descends to $(\Gamma\backslash\mathcal{H}^\pm)^\diamondsuit$; there is also a natural map to Div^1, so we have a map $h : (\Gamma\backslash\mathcal{H}^\pm)^\diamondsuit \to \mathrm{Div}^1 \times [*/\mathbb{H}^\times]$. We can then consider e.g. $h_*\mathcal{O}$, which (for varying Γ) is an analogue of completed cohomology at the infinite place; on the other hand it naturally carries the structure of a $W_{\mathbb{R}} \times \mathbb{H}^\times$-representation. One can give a formula for it, which is sort of a version of Matsushima's formula: if $\mathcal{A}_\Gamma$ is the space of automorphic forms on Γ, viewed as a GL_2-representation, then we should have

$$g_* f^* \mathcal{A}_\Gamma \simeq h_* \mathcal{O}_{(\Gamma\backslash\mathcal{H}^\pm)^\diamondsuit}.$$

(Decomposing into irreducible automorphic representations π recovers the formula above.)

6. Archimedean shtukas

Near the end of the talk, Scholze mentioned the following "suspicion": given a Shimura datum (G, X), one can produce the Shimura variety $\mathrm{Sh}(G, X)$ as well as an inner form G' of G over $\mathbb{R}$ which is compact modulo its center.

For each prime p, in the case where the level $K_p = \mathcal{G}(\mathbb{Z}_p)$ is hyperspecial one can find a $\mathcal{G}(\mathbb{Z}_p)$-torsor $\lim_{K_p' \subset K_p} \mathrm{Sh}_{K_p' K^p}(G, X) \to \mathrm{Sh}_{K_p K^p}(G, X)$, which gives rise to a $\mathcal{G}(\mathbb{Z}_p)$-local system on $\mathrm{Sh}_{K_p K^p}(G, X)$; if (G, X) is of abelian type, this can be associated to various realizations (étale, crystalline, prismatic) on an integral model of $\mathrm{Sh}(G, X)$, which are incarnations of the universal p-adic shtuka on $\mathrm{Sh}(G, X)$. Indeed, we expect that $\mathrm{Sh}(G, X)$ is "secretly" a moduli space for G-shtukas over $\operatorname{Spec}\mathbb{Z}$ (as yet an undefined notion), and so for every p we should expect a universal p-adic shtuka.[2]

At the infinite place, we then also expect to have an "archimedean shtuka" on $\mathrm{Sh}(G, X)$; just as at p this is associated to a torsor for a suitable compact $\mathcal{G}(\mathbb{Z}_p)$, at infinity it should be associated to a $G'(\mathbb{R})$-torsor on (some incarnation of) $\mathrm{Sh}(G, X)$. In the case of GL_2, this would be a $\mathbb{H}^\times$-torsor on the modular curve; this is given by the projection $(\Gamma\backslash\mathcal{H}^\pm)^\diamondsuit \to [*/\mathbb{H}^\times]$.

The remainder of this section will be unfounded speculation about a shtuka interpretation of this suspicion. Let's first briefly review what the p-adic side looks like: a p-adic G-shtuka

[2] When (G, X) is of Hodge type, we can think of these as realizations of the universal abelian variety over $\mathrm{Sh}(G, X)$.

with one leg and level $K \subset G(\mathbb{Q}_p)$ of type $b \in B(G)$ consists of a vector bundle $\mathcal{E}$ on the Fargues–Fontaine curve, a point D in Div^1, an isomorphism $\mathcal{E}^b \simeq \mathcal{E}$ away from D on the curve (and suitable bounded at D), and a K-torsor $\mathcal{F} \subset \mathcal{E}$. Such a shtuka on S can be shown to be equivalent to an untilt $S^\sharp$ together with a p-divisible group with G-structure on $S^\sharp$ which lifts the p-divisible group on $\overline{\mathbb{F}_p}$ corresponding to b and level K-structure; alternatively, slightly unwinding the definition of the Fargues–Fontaine curve, we can think of it as essentially a sort of isocrystal, corresponding to the Dieudonné module of the associated p-divisible group. On the other hand, via prismatic Dieudonné theory we could think of this as the data of a suitable prismatic F-crystal, whose étale realization following [?] gives a K-local system; in particular we get a universal K-local system over the moduli space of shtukas, associated to the universal shtuka.

We no longer have a Frobenius to pull back by, but we can follow along regardless: if $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$ is our replacement for the Fargues–Fontaine curve and $G'(\mathbb{R})$ replaces $\mathcal{G}(\mathbb{Z}_p)$, then an archimedean G-shtuka of type $b \in B(G)$ with one leg should be the data of a G-bundle $\mathcal{E}$ on $\widetilde{\mathbb{P}}^1_{\mathbb{R}}$, a point $D \in \mathrm{Div}^1$, an isomorphism $\mathcal{E}^b \simeq \mathcal{E}$ away from D, and a $G'(\mathbb{R})$-torsor $\mathcal{F}$ mapping to $\mathcal{E}$. (At "infinite level," we should be able to drop this last datum.) We may also want to impose some boundedness conditions as in the p-adic case. One could similarly define spaces of shtukas with more legs, although pinning down the precise definitions might get complicated.

For example, for $G = \mathrm{GL}_2$ and b corresponding to the bundle $\mathcal{O}(1/2)$, working at infinite level if we choose μ such that the modifications are required to be degree 1 injections $\mathcal{E} \to \mathcal{O}(1/2)$ then this forces $\mathcal{E} \simeq \mathcal{O}^{\oplus 2}$, and so we're in the Lubin–Tate situation; and so we might expect that our stack of shtukas is in fact $(\mathcal{H}^\pm)^\diamondsuit$ or some cover of it, with $G' = \mathbb{H}^\times$-torsor given by the infinite-level version $\{\mathcal{O}^{\oplus 2} \hookrightarrow \mathcal{O}(1/2)\}$ (or a base changed version of it).

Now, the main point of interest for p-adic shtukas is that, for one leg, minuscule cocharacters, and suitable groups, their moduli give rise to local Shimura varieties: that is, there exists a suitable adic space over $\mathbb{Q}_p$ whose diamondization recovers $\mathrm{Sht}^1_{G,\mu,b,K}$. These are (at least in good cases) adic generic fibers of Rapoport–Zink spaces $\mathcal{N}^b_{G,K}$. At least conjecturally, one should be able to relate these to Shimura varieties via p-adic uniformization: following Zhang [5] this means that we should be able to write (an incarnation of) $\mathrm{Sh}(G,X)$ as (an explicit quotient of) a product of these Rapoport–Zink spaces with certain Igusa varieties. For b basic, the corresponding Igusa variety is (a finite union of terms of the form) $I(\mathbb{Q})\backslash G(\mathbb{A}_f^p)/K^{p,\infty}$ for an inner form I of G. Our hope would then be that at infinite level, when (G,X) is a Shimura datum with $\mu \in X$ we have $\mathrm{Sht}^1_{G,\mu,b,K_\infty} = (\mathcal{N}^b_{G,K_\infty})^\diamondsuit$ and a stratification of (some version of) $\mathrm{Sh}(G,X)$ by (quotients of) $\mathcal{N}^b_{G,K_\infty} \times \mathrm{Ig}^b$ for an analogous Igusa stack Ig^b, with a similar product description for b basic. But of course we do have such a description of Shimura varieties:

$$\mathrm{Sh}(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times X/K_f,$$

so we recover complex uniformization from this story if $X^\diamondsuit = (G(\mathbb{R})/K_\infty)^\diamondsuit$ is (a suitable quotient of) $\mathrm{Sht}^1_{G,\mu,b,K_\infty}$. There is no restriction to a basic locus; this is compatible with a suggestion of Hartl–Xu ([2, Remark 1.2.1]) that we should think of "shtukas over $\operatorname{Spec}\mathbb{Z}$" as having, in addition to a "varying" leg in $\operatorname{Spec}\mathbb{Z}$ (corresponding to e.g. an abelian variety defined over $\mathbb{F}_q$ with weights of Frobenius related to the slopes of the associated p-divisible group), a fixed leg at infinity (corresponding to the weight condition that the weights of Frobenius act by eigenvalues with archimedean absolute value (a suitable power of) $q^{1/2}$); in particular at the infinite place the weights are all equal, while the slopes of a p-divisible group are only equal in the basic case. In other words, we think of everything as basic at ∞, so that we get the whole Shimura variety rather than just a stratum.

Now p-adic uniformization gives a somewhat stronger statement: rather than just on $\mathbb{C}$- (or $\mathbb{C}_p$-)points, we get a genuine isomorphism of formal schemes/rigid-analytic spaces. We hope that carrying out the above program might yield a similar strengthening of complex uniformization. The sketch above suggests that this should look something like replacing the operation of taking $\mathbb{C}$-points by viewing $\mathrm{Sh}(G, X)$ and X as complex-analytic spaces and taking their complex diamondizations; one should also reinterpret the various groups as constant sheaves of analytic groups. More precisely there should be a fiber product conjecture analogous to Zhang's in general, likely simpler at the archimedean place.

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