Lecture 12: first applications and related rates

Calculus I, section 10
October 25, 2022

We now know how to differentiate pretty much any (differentiable) function we can think of. For the next few weeks, we'll focus on applying this new tool to a variety of situations, some within math, some showing up in other fields or real-life-type problems.

One kind of application is really something we've already seen: the derivative as linear approximation. Instead of talking about approximating e.g. $f(2.01)$ given $f(2)$, though, let's think about a slightly different side of things, which we might call "asymptotic approximations."

For example, a common mistake is to say that for two positive numbers x and y,

$$\sqrt{x+y} = \sqrt{x} + \sqrt{y}.$$

This is *not* true, as can be seen by taking e.g. $x = 16$, $y = 9$; then

$$\sqrt{16+9} = \sqrt{25} = 5, \qquad \text{but} \qquad \sqrt{16} + \sqrt{9} = 4 + 3 = 7.$$

Indeed, there is no good way to expand $\sqrt{x+y}$.

However, if y is very small compared to x, there is a good approximation, using the derivative: if $f(x) = \sqrt{x}$, then

$$f(x+y) \approx f(x) + f'(x)y.$$

In this case, this is

$$\sqrt{x+y} \approx \sqrt{x} + \frac{y}{2\sqrt{x}}.$$

Now, we've seen this sort of approximation before when y is very small, e.g.

$$\sqrt{4.01} \approx \sqrt{4} + \frac{0.01}{2\sqrt{4}} = 2 + \frac{0.01}{4} = 2.0025$$

(the true value is about 2.002498, so this is pretty good). However, it's also a good approximation when x is very large, and y is small only relative to x: for example, while we wouldn't expect this to give a spectacularly good approximation for $x = 4$ and $y = 1$ (though it's still not bad, it gives $\sqrt{5} \approx 2 + \frac{1}{4} = 2.25$, compared to the true value of about 2.2361), for $x = 100$ and $y = 1$ it gives

$$\sqrt{101} \approx \sqrt{100} + \frac{1}{2 \cdot \sqrt{100}} = 10 + \frac{1}{20} = 10.05,$$

compared to the true value of around 10.04988. Indeed, this gives in general

$$\sqrt{x+1} \approx \sqrt{x} + \frac{1}{2\sqrt{x}},$$

which should grow more and more accurate as $x \to \infty$.

Using our knowledge of limits, we can confirm this:

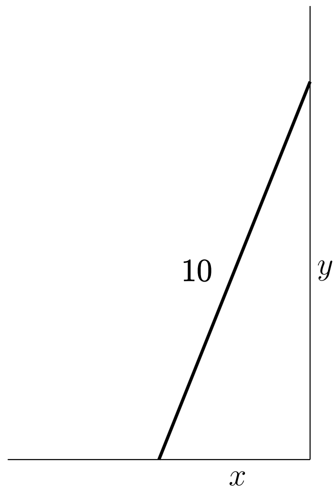
$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x+1} - \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) &= \lim_{x\to\infty} \frac{\sqrt{x+1}\cdot 2\sqrt{x} - \sqrt{x}\cdot 2\sqrt{x} - 1}{2\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{2\sqrt{x^2+x} - 2x - 1}{2\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{2\sqrt{x^2+x} - 2x - 1}{2\sqrt{x}} \cdot \frac{2\sqrt{x^2+x} + 2x + 1}{2\sqrt{x^2+x} + 2x + 1} \\
&= \lim_{x\to\infty} \frac{4(x^2+x) - (2x+1)^2}{2\sqrt{x}(2\sqrt{x^2+x} + 2x + 1} \\
&= \lim_{x\to\infty} \frac{4x^2 + 4x - 4x^2 - 4x - 1}{2\sqrt{x}(2\sqrt{x^2+x} + 2x + 1)} \\
&= \lim_{x\to\infty} \frac{-1}{2\sqrt{x}\cdot(2\sqrt{x^2+x} + 2x + 1)} \\
&= 0.
\end{aligned}$$

If you think about this for a moment, it isn't actually very impressive: already $\sqrt{x+1} - \sqrt{x}$ goes to 0 as $x \to \infty$, since it's approximately $\frac{1}{2\sqrt{x}}$! What we've done is give an approximation to the remainder, which tells us something about *how fast* $\sqrt{x+1} - \sqrt{x}$ goes to 0. One way of phrasing this is: $\sqrt{x+1} - \sqrt{x}$ goes to zero; but if we multiply the whole thing by x (which goes to infinity), will it still go to zero, or now go to infinity, or maybe go to some nonzero number? The above calculation shows that $x(\sqrt{x+1} - \sqrt{x})$ will go to infinity, but $\sqrt{x}(\sqrt{x+1} - \sqrt{x})$ will go to a real number, namely $\frac{1}{2}$. Understanding functions like this, and how they change when x is perturbed by a relatively small amount, is one of the most common day-to-day applications of derivatives within pure math. Your homework for this week will have a similar problem, involving logarithms instead of square roots.

Here's another problem, towards a different kind of application, and without which no calculus course would be complete: imagine a 10-foot ladder is leaning against a wall, and the base of the ladder is sliding away from the wall at a rate of 2 feet per second. How fast is the top of the ladder moving downwards?[1]

We start by drawing a picture:

[1] We assume that the top of the ladder is sliding against the wall and never comes off it; in real life, it would eventually come off the wall, and becomes a more complicated physics problem.



We know that $\frac{dx}{dt} = 2$ (in feet per second), and we want to find $\frac{dy}{dt}$. How can we do this? Well, we have a relationship between x and y:

$$x^2 + y^2 = 10^2 = 100.$$

Therefore it's natural to try differentiating it, now not with respect to x but with respect to t, on which both x and y depend:

$$\frac{d}{dt}x^2 + \frac{d}{dt}y^2 = \frac{d}{dt}100.$$

The right-hand side is simply 0, as 100 is constant; we evaluate the left-hand side by the chain rule,

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0.$$

Now we know $\frac{dx}{dt} = 2$, so we can solve for $\frac{dy}{dt}$:

$$\frac{dy}{dt} = -\frac{4x}{2y}.$$

Since x and y are under the constraint above, $x^2 + y^2 = 100$, we can find the speed at any given position. For example, if we imagine that at $t = 0$ the ladder is completely vertical, i.e. $y = 10$ and $x = 0$, then even though the horizontal velocity is constant ($\frac{dx}{dt} = 0$) the vertical velocity is initially 0. Similarly, at the point where the ladder hits the ground, i.e. is horizontal, we have $x = 10$ and $y = 0$, and we see that the vertical speed actually goes to infinity here! (This is one way of showing that the situation described here is not quite physically possible: either the horizontal velocity of the ladder will not actually be constant or the top of the ladder will come off the wall before hitting the ground.)

The method we use here is called *related rates*. It is very similar to implicit differentiation: we have a relationship between a few quantities, and instead of solving for one of them in terms of the others and then differentiating, we first differentiate and then simplify. The difference is that instead of having a relationship between x and y and then differentiate with respect to x, here both x and y (and potentially other quantities) depend on another variable, which in practice is frequently t for time. We know some of the rates of change of these quantities, but not all; we use the relationship between these rates, which we get from differentiating the relationships, to find whichever rate we're actually looking for. This is a frequent feature in applications.

Another example is as follows. Let's imagine that a drop of water finds itself in space. In space, the drop becomes a perfect sphere under its own gravity, say of radius r. It also immediately begins to boil (or, depending on its temperature, evaporate). The rate at which water evaporates into space is proportionate to the surface area of the sphere. In other words, if m is the mass of our drop and A is its surface area, then

$$\frac{dm}{dt} = cA$$

for some constant c.[2] How fast is the radius r changing?

We have the following relationships, from the geometry of spheres:

$$m = \frac{4}{3}\pi r^3$$

and

$$A = 4\pi r^2.$$

Therefore

$$\frac{dm}{dt} = \frac{d}{dt}\frac{4}{3}\pi r^3 = 4\pi r^2 \cdot \frac{dr}{dt} = A \cdot \frac{dr}{dt}.$$

So the result is surprisingly simple:

$$\frac{dm}{dt} = cA = A \cdot \frac{dr}{dt},$$

so

$$\frac{dr}{dt} = c$$

is actually constant! In particular, if the drop starts at radius r_0, then it is changing at a constant rate of c and so after $-\frac{r_0}{c}$ seconds it has completely evaporated.[3]

Let's do one more example: the ideal gas law states that given n moles of an (idealized) gas at temperature T, pressure P, and volume V, these must be related by

$$PV = nRT,$$

where R is a certain constant (about 8.314 kPa $\cdot$ L/K). If we have one mole of gas held in a rigid one-liter container, if we increase the temperature at a constant rate of 2 degrees Kelvin per second, what is the rate of change of the pressure?

[2] Finding or even guessing at this constant appears to be a surprisingly tricky physics problem. Note that it should be negative, since the mass is decreasing and the surface area is positive.

[3] We're implicitly using some ideas of integration here, so don't worry if this last line doesn't make sense.

This one is easy: we have $V = 1$ and $n = 1$, so the equation is just $P = RT$ and so $\frac{dP}{dt} = R\frac{dT}{dt} = 2R$. A more difficult question would be if we instead fix the temperature and vary the volume: if the volume is decreasing at a fixed rate of .1 liters per second, how fast is the pressure increasing?

It might be tempting to say it is also increasing at a fixed rate, but this is very much not the case. Instead, we have $PV = RT$ and so by the product rule

$$\frac{dP}{dt}V + P\frac{dV}{dt} = 0,$$

so since V is decreasing at .1 liters per second

$$\frac{dP}{dt}V - 0.1 \cdot P = 0,$$

i.e.

$$\frac{dP}{dt} = 0.1 \cdot \frac{P}{V}.$$

Thus the rate of change of P is proportionate to P and inversely proportionate to V. Since P and V are related, we could rewrite this either as $V = \frac{RT}{P}$ and so

$$\frac{dP}{dt} = 0.1 \cdot \frac{P}{RT/P} = \frac{0.1}{RT} \cdot P^2$$

or $P = \frac{RT}{V}$ and so

$$\frac{dP}{dt} = 0.1 \cdot \frac{RT/V}{V} = 0.1 \cdot RT \cdot \frac{1}{V^2}.$$

This lets us find the rate of change in pressure as a function of pressure and volume; we can see that as the volume goes to 0, the rate of change in pressure tends to infinity, even though the rate of change in volume is constant. (This is not surprising physically: if there is no space for the gas to be in, the pressure will blow up to infinity (and most likely the container will physically explode)).

This is also an example of a *differential equation*: the derivative of P is related back to P. We won't solve these in this class, but it is possible using the methods of calculus; if we solved this one, we could get a formula for pressure telling us that if at time $t = 0$ the pressure is P_0, then

$$P(t) = \frac{10P_0RT}{10RT - P_0 \cdot t}.$$

You can check that this satisfies the differential equation above.

Next time we'll start talking about another big class of applications: using derivatives to find the extrema (maxima, minima, and other critical points) of functions, which answers one of the first questions we asked in this class and has many applications which aren't at first look relevant to calculus.