Topology of $\mathbb{G}_m$-equivariant morphisms

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Abstract

Let $f : X \to Y$ be a $\mathbb{G}_m$-equivariant morphism of separated schemes over a base. Suppose that "limits exist" in X and the action of $\mathbb{G}_m$ is "contracting" (explained below, based on [HZ23, Appendix B]). Then we observe that, under mild assumptions, the topology of the fiber over the locus of fixed points $X \times_Y Y^{\mathbb{G}_m} \to Y^{\mathbb{G}_m}$ implies global properties about the morphism $X \to Y$.

Let S be a locally Noetherian base scheme. Recall the following:

Definition 1 ([HZ23, Def. B1])**.** *Let Y be a separated scheme locally finite type over S. An action of $\mathbb{G}_m$ on Y has zero limits if for all algebraically closed fields k and k-points $f : Spec(k) \to Y$, there exists a $\mathbb{G}_m$-equivariant morphism $\widetilde{f} : \mathbb{A}^1_k \to Y$ such that $\widetilde{f}(1) = f$.*

We say that an action of $\mathbb{G}_m$ on Y is contracting if for all discrete valuation rings R and morphisms $f : Spec(R) \to Y$, there exists a $\mathbb{G}_m$-equivariant morphism $\widetilde{f} : \mathbb{A}^1_R \to Y$ such that $\widetilde{f}(1) = f$.

Recall that for any scheme Y separated locally of finite type over S equipped with a $\mathbb{G}_m$-action, we can define a functor Y^+ from S-schemes to sets that sends $T \to S$ to the set of $\mathbb{G}_m$-equivariant morphism of T-schemes $\mathbb{A}^1_T \to Y_T$. This is represented by an algebraic space locally of finite type over S [HL22, Prop. 1.4.1]. There are natural morphisms $\mathrm{ev}_0 : Y^+ \to Y$ and $\mathrm{ev}_1 : Y^+ \to Y$ defined by evaluating at $0 \in \mathbb{A}^1$ and $1 \in \mathbb{A}^1$ respectively. The morphism $\mathrm{ev}_1 : Y^+ \to Y$ is a monomorphism of finite type, and so it follows that Y^+ is also a separated scheme locally of finite type over S. The morphism $\mathrm{ev}_0 : Y^+ \to Y$ factors through the closed subscheme $Y^{\mathbb{G}_m} \subset Y$ of $\mathbb{G}_m$-equivariant points (denote by Y^0 in [HL22, Prop. 1.4.1]).

In terms of the morphism $\mathrm{ev}_1 : Y^+ \to Y$, we have the following interpretations:

(1) The $\mathbb{G}_m$-action has zero limits if and only if $\mathrm{ev}_1 : Y^+ \to Y$ is surjective.

(2) The $\mathbb{G}_m$-action is contracting if and only if $\mathrm{ev}_1 : Y^+ \to Y$ is a surjective closed immersion.

In particular, if Y is reduced and the $\mathbb{G}_m$-action is contracting, then $Y^+ = Y$. Moreover, we see that if the $\mathbb{G}_m$-action is contracting, then for all reduced schemes T and morphisms $f : T \to Y$, there is a $\mathbb{G}_m$-equivariant extension $\widetilde{f} : \mathbb{A}^1_T \to Y_T$.

Context 2. *Let $f : X \to Y$ be a $\mathbb{G}_m$-equivariant morphism of separated locally of finite type S-schemes. We suppose that f is of finite type, the $\mathbb{G}_m$-action on X has zero limits, and the $\mathbb{G}_m$-action on Y is contracting.*

Remarkably, under the assumptions in Context 2, some global topological properties of the morphism f are controlled by the fiber over the fixed points $X \times_Y Y^{\mathbb{G}_m} \to Y^{\mathbb{G}_m}$. An example of such property is properness.

Proposition 3 ([HZ23, Prop. B5])**.** *In Context 2, suppose that the fiber product $X \times_Y Y^{\mathbb{G}_m} \to Y^{\mathbb{G}_m}$ is a proper morphism. Then, f is a proper morphism.*

Another topological property controlled by the fiber $X \times_Y Y^{\mathbb{G}_m} \to Y^{\mathbb{G}_m}$ is (dis)connectedness.

Proposition 4. *In Context 2, suppose that f is proper. Then, the fibers of $X \to S$ are geometrically connected if and only if the fibers of $X \times_Y Y^{\mathbb{G}_m} \to S$ are geometrically connected.*

Proof. Without loss of generality we can base-change to a geometric point of S and assume that $S = \operatorname{Spec}(k)$ for some algebraically closed field k. If $X \times_Y Y^{\mathbb{G}_m}$ is connected, then it follows that X is also connected, since every k-point of X can be connected to the fiber $X \times_Y Y^{\mathbb{G}_m}$ via a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_k \to X$ (here we only use the hypothesis that X has zero limits).

For the converse, suppose that we can write $X \times_Y Y^{\mathbb{G}_m} = U \sqcup V$, where U, V are open and closed subschemes. Notice that we have a morphism $X^+ \xrightarrow{\mathrm{ev}_0} X^{\mathbb{G}_m} \hookrightarrow X \times_Y Y^{\mathbb{G}_m}$. We denote by $X^+ = X^+_U \sqcup X^+_V$ the decomposition induced by the preimages of U and V. Notice that by properness it follows that neither X^+_U nor X^+_V are empty. Consider the surjective monomorphism $\mathrm{ev}_1 : X^+ \to X$ (it is surjective because X has zero limits). We claim that the images of $\mathrm{ev}_1 : X^+_U \to X$ and $\mathrm{ev}_1 : X^+_V \to X$ are closed. This implies that X is disconnected, as desired.

To show the claim, we may replace Y with its reduced subscheme, and hence assume that $Y^+ = Y$. By symmetry, it suffices to show that the image of X^+_U is closed. Suppose for the sake of contradiction that the image of X^+_U is not closed. By [PPN17, Lemma 2.4], it follows that there exist

(1) A $\mathbb{G}_m$-fixed k-point x_V in the image of X^+_V (we think of x_V as a point in $V \subset X \times_Y Y^{\mathbb{G}_m} \subset X$).

(2) A smooth connected curve $i : C \hookrightarrow X$ equipped with a k-point $p \in C(k)$ such that $i(C \setminus p)$ is contained in the image of X^+_U and $i(p) = x$.

Moreover, after perhaps shrinking the curve and passing to a cover, we may assume that the morphism $C \setminus p \to X$ factors through X^+_U.

Since $Y^+ = Y$, the composition $C \xrightarrow{i} X \to Y$ can be (uniquely) completed to a morphism $\Sigma := C \times \mathbb{A}^1_k \to Y$. Since $i : C \setminus p \to X$ factors through X^+_U, the restriction to the open $(C \setminus p) \times \mathbb{A}^1_k$ lifts to an equivariant morphism $(C \setminus p) \times \mathbb{A}^1_k \to X$. Furthermore, using the $\mathbb{G}_m$-action we can also lift over the open $C \times (\mathbb{A}^1_k \setminus 0)$, and so we get a $\mathbb{G}_m$-equivariant morphism $\Sigma \setminus p \times 0 \to X$ of schemes over Y. Since $X \to Y$ is proper, using resolution of 2-dimensional schemes we may resolve the indeterminacies of this morphism by blowing up over the point $p \times 0$ in order to obtain a smooth $\mathbb{G}_m$-surface $\widetilde{\Sigma} \to \Sigma$ and a $\mathbb{G}_m$-equivariant morphism of Y-schemes $h : \widetilde{\Sigma} \to X$. Consider the strict transforms $\widetilde{D}_1, \widetilde{D}_2 \subset \widetilde{\Sigma}$ of $D_1 = p \times \mathbb{A}^1_k$ and $D_2 = C \times 0$ respectively. By construction, the morphism $h : \widetilde{D}_1 \to X$ factors through the $\mathbb{G}_m$-fixed point $x = i(p)$ in $V \subset X \times_Y Y^{\mathbb{G}_m}$. On the other hand, for the morphism $h : \widetilde{D}_1 \cong C \times 0 \to X \times_Y Y^{\mathbb{G}_m} \subset X$, notice that by assumption the restriction $h : (C \setminus p) \times 0 \to X$ is a limit of the original morphism $i : C \setminus p \to X_U^+ \to X$. By the definition of X_U^+, it follows that the image of $h : (C \setminus p) \times 0 \to X \times_Y Y^{\mathbb{G}_m}$ lies inside U. Since U is closed, the image of $C \cong \widetilde{D}_1 \xrightarrow{h} X \times_Y Y^{\mathbb{G}_m}$ also lies in U, we denote x_U the image of $p \times 0 \in C \cong \widetilde{D}_1$. Notice that the exceptional divisor $E = \widetilde{\Sigma}_{p\times 0} \subset \widetilde{\Sigma}$ is connected, and its image lies in $X \times_Y Y^{\mathbb{G}_m}$ and is a connected curve joining the points $x_V \in V$ and $x_U \in U$. This contradicts the fact that $X \times_Y Y^{\mathbb{G}_m} = U \sqcup V$ was a decomposition into disjoint open and closed subschemes. This contradiction concludes the proof that the image of X_U^+ is closed. □

In [HZ23] we offered an application of Proposition 3 to prove the properness of the p-curvature morphism for t-connections on G-bundles over a curve. On the other hand, the statement of Proposition 4 was an attempt to extract the topological content of the arguments in [PPN17]; it is not surprising then that Proposition 4 can be used to easily show [PPN17, Thm. 1.1] and more generally [HH22, Thm. 1.1].

We encourage the interested reader to find more global properties of the morphism $f : X \to Y$ that could possibly be determined by the fiber $X \times_Y Y^{\mathbb{G}_m} \to Y^{\mathbb{G}_m}$.

References

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