Normal and tangent spaces

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These are notes from a conversation with Andrés Fernandez Herrero.

Let $\mathcal{X}$ be an algebraic stack, locally of finite type over a filed k. It is possible that more assumptions on $\mathcal{X}$ are necessary for the results that follow to hold. Let $x \in \mathcal{X}(k)$ be a closed point and $i\colon BG_x \to \mathcal{X}$ be the residual gerbe, which is a closed immersion with ideal sheaf $\mathcal{J}$. The *normal space to* x is defined to be $N_x := (i^*\mathcal{J})^\vee$ as a G_x representation. The tangent space $T_{\mathcal{X},x}$ at x is defined as usual as isomorphism classes of objects of the fibre (groupoid) at x of $\mathcal{X}(\mathrm{Spec}(k[\epsilon]/(\epsilon^2)) \to \mathcal{X}(\mathrm{Spec}(k))$, with the induced structure of G_x-representation. By deformation theory, we can see the tangent space using sheaves of differentials:

Fact 1. $T_{\mathcal{X},x} = H^0(i^*\mathbb{L}_{\mathcal{X}/k})^\vee$.

We will setch a proof of the following.

Proposition 2. *If G_x is smooth, then $T_{\mathcal{X},x} = N_x$.*

Proposition 3. *$\mathbb{L}_{BG_x/k}$ is concentrated in degree* 1.

Proof. The sequence of morphisms

$$\mathrm{Spec}\, k \xrightarrow{c} BG_x \longrightarrow \mathrm{Spec}\, k$$

induces a distinguished triangle

$$c^*\mathbb{L}_{BG_x/k} \longrightarrow \mathbb{L}_{k/k} \longrightarrow \mathbb{L}_{k/BG_x} \xrightarrow{+1} .$$

Thus $c^*\mathbb{L}_{BG_x/k} = \mathbb{L}_{k/BG_x}[-1]$, which is concentrated in degree 1, $\mathrm{Spec}(k) \to BG_x$ being smooth and representable. By flatness of c, we have $H^i(c^*\mathbb{L}_{BG_x/k}) = c^*H^i(\mathbb{L}_{BG_x/k})$, since c^* is exact. Now, applying descent for the smooth cover c to the sheaves $H^i(\mathbb{L}_{BG_x/k})$, we get that $\mathbb{L}_{BG_x/k}$ is concentrated in degree 1 as well. □

Proof of Proposition 2. From the sequence

$$BG_x \xrightarrow{i} \mathcal{X} \longrightarrow \mathrm{Spec}\, k$$

we get an exact triangle

$$i^*\mathbb{L}_{\mathcal{X}/k} \longrightarrow \mathbb{L}_{BG_x/k} \longrightarrow \mathbb{L}_{BG_x/\mathcal{X}} \xrightarrow{+1} .$$

Since i is a closed immersion, we know $H^i(\mathbb{L}_{BG_x/\mathcal{X}}) = \begin{cases} 0, & i > -1, \\ i^*\mathcal{J}, & i = -1 \end{cases}$.

We can write an exact sequence

$$0 = H^{-1}(\mathbb{L}_{BG_x/k}) \longrightarrow H^{-1}(\mathbb{L}_{BG_x/\mathcal{X}}) \longrightarrow H^0(i^*\mathbb{L}_{\mathcal{X}/k}) \longrightarrow H^0(\mathbb{L}_{BG_x/k}) = 0$$

giving an isomorphism $T_{\mathcal{X},x} = N_x$. □