Some remarks on equivariant sheaves

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Abstract

This short note presents in an explicit manner the equivariant structure of the cotangent sheaf on any scheme with an arbitrary group action. Also, we discuss why the group should act on the cohomology of any equivariant sheaf.

1 Equivariant sheaves

Fix a scheme S. For this note we will work on the category of schemes over S. All schemes will have an implicit structure map to S. All unlabeled products are to be interpreted as fiber products over S.

Let X be a S-scheme. Let G be a group scheme over S. Let us denote by $m : G \times G \to G$ the multiplication morphism. Let $a : G \times X \to X$ be a left action of G on X. Recall that this means that for all schemes T the induced map $G(T) \times X(T) \to X(T)$ is a left action of the group $G(T)$ on the set $X(T)$.

Let us set up the notation for the groupoid scheme associated to the action a. This is just the groupoid in schemes whose stackification is the quotient stack $[G\backslash X]$. We have a diagram

$$G \times G \times X \overset{p_{12}}{\underset{p_{13}}{\overset{\xrightarrow{p_{23}}}{\rightrightarrows}}} G \times X \underset{p_2}{\overset{p_1}{\rightrightarrows}} X$$

The morphisms depicted above are defined as follows (on T-points)

$p_1(g, x) = a(g, x)$.

$p_2(g, x) = x$.

$p_{12}(g, h, x) = (g, a(h, x)$.

$p_{23}(g, h, x) = (g, x)$.

$p_{13}(g, h, x) = (gh, x)$.

In addition, define

$s_1 = p_1 p_{12} = p_1 p_{13}$.

$s_2 = p_2 p_{12} = p_1 p_{23}.$

$s_3 = p_2 p_{13} = p_2 p_{23}.$

An equivariant sheaf is just a quasicoherent sheaf on this groupoid scheme. Let's spell this out.

Definition 1.1. *Let $\mathcal{F}$ be a quasicoherent sheaf on X. A G-equivariant structure on $\mathcal{F}$ consists of the datum of an isomorphism $\varphi : p_1^*\mathcal{F} \to p_2^*\mathcal{F}$. This isomorphism is required to make the following diagram commute*

$$\begin{tikzcd}
s_1^*\mathcal{F} \arrow[r, "\sim"] \arrow[d, "\sim"] & p_{13}^*p_1^*\mathcal{F} \arrow[r, "p_{13}^*\varphi"] & p_{13}^*p_2^*\mathcal{F} \arrow[r, "\sim"] & s_3^*\mathcal{F} \\
p_{12}^*p_1^*\mathcal{F} \arrow[r, "p_{12}^*\varphi"] & p_{12}^*p_2^*\mathcal{F} \arrow[r, "\sim"] & p_{23}^*p_1^*\mathcal{F} \arrow[r, "p_{23}^*\varphi"] & p_{23}^*p_2^*\mathcal{F} \arrow[u, "\sim"']
\end{tikzcd}$$

Remark 1.2. *Basically, the condition above says that the two obvious ways to go from $s_1^*\mathcal{F}$ to $s_3^*\mathcal{F}$ coincide.*

2 Example: the cotangent bundle

Let X and G be as in the previous section. There is always a natural G-equivariant structure on the sheaf of Kähler differentials $\Omega^1_{X/S}$. Let us describe how this works.

Recall that for any maps of S-schemes $f : X \to Y$, there is an associated morphism of cotangent sheaves $df : f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$. The operation $d \mapsto df$ behaves well under composition; the reader can be trusted with determining the right commutative diagram for compositions.

By working affine locally, one can also see that $\Omega^1_{-/S}$ is well-behaved with respect to products. Let's spell this out for the product $G \times X$. Notice that there are two projections $\mathrm{pr}_1 : G \times X \to G$ and $\mathrm{pr}_2 : G \times X \to X$. These yields two maps $d\mathrm{pr}_1 : \mathrm{pr}_1^*\Omega^1_{G/S} \to \Omega^1_{G\times X/S}$ and $d\mathrm{pr}_2 : \mathrm{pr}_2^*\Omega^1_{X/S} \to \Omega^1_{G\times X/S}$. Both of these are injective, so we can identify $\mathrm{pr}_1^*\Omega^1_{G/S}$ and $\mathrm{pr}_2^*\Omega^1_{X/S}$ with their respective images in $\Omega^1_{G\times X/S}$. It turns out that the images span $\Omega^1_{G\times X/S}$ and do not intersect. So we get a direct sum decomposition $\Omega^1_{G\times X/S} = \mathrm{pr}_1^*\Omega^1_{G/S} \oplus \mathrm{pr}_2^*\Omega^1_{X/S}$. We will denote by π_1, π_2 the projections from $\Omega^1_{G\times X/S}$ onto these direct summands.

The G-equivariant structure on $\Omega^1_{X/S}$ is given by the composition

$$p_1^*\Omega^1_{X/S} \xrightarrow{dp_1} \Omega^1_{G\times X} \xrightarrow{\pi_2} \mathrm{pr}_2^*\Omega^1_{X/S} = p_2^*\Omega^1_{X/S}$$

A section $\omega \in H^0(X, \Omega^1_{X/S})$ is called G-invariant if $\pi_2(p_1^*\omega) = p_2^*\omega$. For example, a 1-form ω on G is bi-invariant exactly when $\pi_1(m^*\omega) = \mathrm{pr}_1^*\omega$ and $\pi_2(m^*\omega) = \mathrm{pr}_2^*\omega$. In other words, $m^*\omega = \mathrm{pr}_1^*\omega + \mathrm{pr}_2^*\omega$. This is something that one sees early on when proving results about elliptic curves using the invariant differential.

3 Action on cohomology

Let X and G be as in the previous sections. We will suppose that the structure morphism $\pi : X \to S$ is quasicompact and quasiseparated.

Let $\mathcal{K}$ be a quasicoherent sheaf on S. We want to make clear what we mean by a representation of the group G on $\mathcal{K}$. When G is affine over S, such representation is determined by giving $\mathcal{K}$ the structure of a comodule over the $\mathcal{O}_S$-Hopf algebra $\mathcal{O}_G$. We use a functorial approach to generalize to nonaffine groups.

Definition 3.1. *Let $\mathcal{K}$ be a quasicoherent sheaf on S. We define $Aut(\mathcal{K})$ to be the functor from S-schemes into groups given as follows. For any $f : T \to S$, we set*

$$Aut(\mathcal{K})\,(T) := \{\, \mathcal{O}_T\text{-automorphisms of the sheaf } f^*\mathcal{K} \ \}$$

Remark 3.2. *One needs to use the canonical isomorphism between the pullback of a composition and the composition of pullbacks in order to realize the functoriality of $Aut(\mathcal{K})$.*

Definition 3.3. *Let $\mathcal{K}$ be a quasicoherent sheaf on S. A G-representation on $\mathcal{K}$ is a natural transformation $\tau : G \to Aut(\mathcal{K})$ of group functors.*

Remark 3.4. *The Yoneda lemma tells us that τ is the same as giving an element in $Aut(\mathcal{K})(G)$. This is the same as equipping $\mathcal{K}$ with the structure of a G-equivariant sheaf (here G acts trivially on S).*

Suppose now that the structure homomorphism $f : G \to S$ is flat. Let $\mathcal{F}$ be a G-equivariant sheaf on X. Then, the derived pushforwards $R^i\pi_*\mathcal{F}$ acquire the structure of a G-representation for all $i \geq 0$. Let us see how this works. We have to give a natural transformation $\tau : G \to \mathrm{Aut}(R^i\pi_*\mathcal{F})$. By the Yoneda Lemma, this amounts to giving an element of $\mathrm{Aut}(R^i\pi_*\mathcal{F})\,(G)$. So we are looking for an automorphism of the sheaf $f^*R^i\pi_*\mathcal{F}$ on G. Notice that we have the following Cartesian diagrams

$$\begin{array}{ccc} G \times X & \xrightarrow{p_1} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi} \\ G & \xrightarrow{f} & S \end{array}$$

$$\begin{array}{ccc} G \times X & \xrightarrow{p_2} & X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \pi} \\ G & \xrightarrow{f} & S \end{array}$$

Since π is quasicompact and quasi-separated, flat base-change tells us that there are canonical isomorphisms $f^*R^i\pi_*\mathcal{F} \xrightarrow{\sim} R^i\mathrm{pr}_{1,*}p_1^*\mathcal{F}$ and $f^*R^i\pi_*\mathcal{F} \xrightarrow{\sim} R^i\mathrm{pr}_{1,*}p_2^*\mathcal{F}$. The automorphism giving the sought-after G-representation is given by the composition

$$f^*R^i\pi_*\mathcal{F} \xrightarrow{\sim} R^i\mathrm{pr}_{1,*}p_1^*\mathcal{F} \xrightarrow{R^i\mathrm{pr}_{1,*}\varphi} R^i\mathrm{pr}_{1,*}p_2^*\mathcal{F} \xrightarrow{\sim} f^*R^i\pi_*\mathcal{F}$$

where $\varphi : p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ is the isomorphism coming from the G-equivariant structure on $\mathcal{F}$.

Note that we crucially used the fact that G is flat in the description above. If G is not flat, then derived considerations kick in, and one can't expect an action of G on each higher pushforward. If $\mathcal{F}$ is S-flat, then one can use a good Cech hypercover to describe a flat representative $C^\bullet$ for the derived pushforward $R\pi_*\mathcal{F}$ (if π is separated it suffices to take a Zariski cover of X consisting of sets that are affine over S). Then one can define in an obvious way a group functor $\mathrm{Aut}(C^\bullet)$ of automorphisms of the complex $C^\bullet$. We can use the base-change of the hypercover to G in order to compute the derived pushforward $R\mathrm{pr}_{1,*}$. Since $\mathcal{F}$ is flat, $f^*C^\bullet$ is canonically isomorphic to $R{\mathrm{pr}_1}_* p_1^*\mathcal{F}$ when the latter is computed using the base-changed hypercover. The same goes for $R{\mathrm{pr}_1}_* p_2^*\mathcal{F}$. So we can use a similar description as above to obtain an action of G on the complex $C^\bullet$ of quasicoherent sheaves on S.

If neither G nor $\mathcal{F}$ are flat over S, then one has to deal with derived pullbacks. In this generality it useful to use the derived category. We want to define an action of G on an object of $D(\mathrm{QCoh}(S))$. Derived algebraic geometry seems the most natural context to answer these kinds of questions. Notice that the original argument we gave for G flat works verbatim once we take derived fiber products. This in turn tells us how to define the concept of G-equivariant structure in the context of derived algebraic geometry (an isomorphism of the two pullbacks of $\mathcal{F}$ to the derived fiber product $G \times^{\mathbb{L}} X$ satisfying an analogous cocycle condition). This is the natural set-up if one wants to obtain G-representations on the cohomology of equivariant sheaves when G is not flat.

Remark 3.5. *One can translate this concrete picture with group actions on sheaves into the language of stacks. What we are doing is computing the pushforward of the morphism $[G\backslash X] \to [G\backslash S]$ induced on quotient stacks (these are algebraic stacks when G is smooth). We are just spelling out what is the G-equivariant strucure on the pushforward.*

4 Example: Hoschild Cohomology

Let S be a scheme and $f : G \to S$ be a smooth qcqs group scheme over S. Let $\mathcal{F}$ be a quasicoherent sheaf on S.

Definition 4.1. *$QCoh^G(S)$ is defined to be the abelian category of quasicoherent sheaves on S equipped with a G-action. Here the morphisms are required to intertwine the G-action.*

There is a forgetful functor $F : \mathrm{QCoh}^G(S) \to \mathrm{QCoh}(S)$ that is exact and reflects isomorphisms. It turns out that F admits a left adjoint $H : \mathrm{QCoh}(S) \to \mathrm{QCoh}^G(S)$. Let's describe H. Notice that the quasicoherent sheaf $f_*\mathcal{O}_G$ is equipped with an action induced by right translations. Here one needs to use flat base-change in order to describe the action, the diagrams are the same as in the previous section.

For any $\mathcal{F} \in \mathrm{QCoh}(S)$, we define $H(\mathcal{F}) = \mathcal{F} \otimes f_*\mathcal{O}_G$. We equip this sheaf with a G-action by letting G act on the second factor of the tensor product.

We can use the adjunction (F, H) to see that $\mathrm{QCoh}^G(S)$ has enough injectives. Just use the fact that $\mathrm{QCoh}(S)$ has enough injectives and R preserves injectives. Consider the functor $\mathrm{Inv} : \mathrm{QCoh}^G(S) \to \mathrm{QCoh}(S)$ that takes G-invariants of a given quasicoherent sheaf (you can just take $G(U)$ invariants of global sections for every $U \subset S$ open). For any $\mathcal{F} \in \mathrm{QCoh}^G(S)$, the right derived functors $R^i\mathrm{Inv}\mathcal{F}$ are called the Hoschild cohomology of the quasicoherent G-module $\mathcal{F}$. They are the scheme-theoretic analogues of abstract group cohomology. This cohomology groups become very useful in the theory of algebraic groups, because they arise naturally as obstructions to splitting exact sequences or lifting homomorphisms of groups. One example is the proof of existence of Levi subgroups for algebraic groups over a field of char 0. See [Con14, B.2] or [CGP15] for more applications.

Remark 4.2. *Equip S with the trivial G action. Let $[G\backslash S]$ be the corresponding quotient stack. We have a smooth covering map $g : S \to [G\backslash S]$ and a structure map $\pi : [G\backslash S]$. $QCoh^G(S)$ is just the category of quasicoherent sheaves on $[G\backslash S]$. The forgetful functor F is the pullback g^*. The right adjoint H is just the pushforward g_*. Taking invariants corresponds to applying the pushforward π_*.*

Since the cotangent complex of $[G\backslash S]$ is given by $\pi_*\Omega^1_{G/S}[1]$, it is not surprising that the Hoschild cohomology of $Lie(G)^\vee$ appears as an obstruction to lifting problems in [Con14, B.2].

We would like to concretely compute the Hoschild cohomology of a sheaf. This works especially well when G is relatively affine over S, let us restrict to this case. Since everything is Zariski local on the target, we might as well assume that S is affine, say $S = \mathrm{Spec}\, A$. Notice that taking invariants can be expressed as $\mathrm{Inv}(-) = \mathrm{Hom}_{\mathrm{QCoh}^G(S)}(\mathcal{O}_S, -)$, where $\mathcal{O}_S$ is equipped with the trivial G-action. Here the right-hand side $\mathrm{Hom}_{\mathrm{QCoh}^G(S)}(\mathcal{O}_S, -)$ is naturally a A-module, so it is indeed a quasicoherent sheaf on S. We will imitate the computation of abstract group cohomology.

Since F reflects isomorphisms, we can use the monad coming from the adjunction (F, H) in order to obtain a Bar resolution of $\mathcal{O}_S$

$$... \longrightarrow \mathcal{O}_G^{\oplus 3} \longrightarrow \mathcal{O}_G^{\oplus 2} \longrightarrow \mathcal{O}_G \longrightarrow \mathcal{O}_S$$

(We are ommiting the pushforward symbols because everything is affine, so pushing forward is just viewing everything as A-modules). Let $\mathcal{F} \in \mathrm{QCoh}^G(S)$. Applying $\mathrm{Hom}_{\mathrm{QCoh}^G(S)}(-, \mathcal{F})$ we obtain a complex of A-modules

$$\mathrm{Inv}(\mathcal{F}) \longrightarrow C^0(G, \mathcal{F}) \longrightarrow C^1(G, \mathcal{F}) \longrightarrow ...$$

Here $C^n(G, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Set}}(\, G(S)^n,\, \mathcal{F}\,)$ (the reasoning is the same as for group cohomology, just look at sections in each open and use the actions we defined). The description of the differential d is the same as for abstract group cohomology. We claim that the cohomology of the complex

$$C^0(G, \mathcal{F}) \longrightarrow C^1(G, \mathcal{F}) \longrightarrow C^2(G, \mathcal{F})...$$

computes $R^i\mathrm{Inv}(\mathcal{F})$. This follows if we can show that $\mathcal{O}_G^{\oplus n}$ are projective objects in $\mathrm{QCoh}^G(S)$. But this is clear, because we have seen that $\mathrm{Hom}_{\mathrm{QCoh}^G(S)}(\mathcal{O}_G^{\oplus n}, -) = \mathrm{Hom}_{\mathrm{Set}}(G(S)^n, -)$ and the latter functor is clearly exact.

References

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 26 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, second edition, 2015.

[Con14] Brian Conrad. Reductive group schemes. In *Autour des schémas en groupes. Vol. I*, volume 42/43 of *Panor. Synthèses*, pages 93–444. Soc. Math. France, Paris, 2014.