Topology of residual gerbes

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This document was written to answer a question of Andres Ibanez Nunez.

Definition 0.1. *A monomorphism $f : \mathfrak{X} \to \mathfrak{Y}$ of algebraic stacks is said to be ust if for any morphism of algebraic stacks $\mathfrak{Z} \to \mathfrak{Y}$, the topological space of the base-change $|\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Z}| \subset |\mathfrak{Z}|$ has the subspace topology from $|\mathfrak{Z}|$.*

Lemma 0.2. *The following statements hold.*

(1) Open immersions are ust monomorphisms.

(2) Closed immersions are ust monomorphisms.

(3) The composition of ust monomorphisms is again a ust monomorphism.

(4) If $f : \mathfrak{X} \to \mathfrak{Y}$ is a ust monomorphism and $\mathfrak{Z} \to \mathfrak{Y}$ is a morphism of algebraic stacks, then the base-change $f_{\mathfrak{Z}} : \mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Z}$ is a ust monomorphism.

Proof. These all follow directly from the definition. □

Proposition 0.3. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a monomorphism of algebraic stacks. In order to prove that f is ust, it suffices to check the subspace topology property for base-changes of morphisms $Z \to \mathfrak{Y}$ where Z is a scheme.*

Proof. Let $\mathfrak{Z} \to \mathfrak{Y}$ be a morphism of algebraic stacks. We need to show that any open substack U of $\mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X}$ is the preimage of an open substack of $\mathfrak{Z}$ under the inclusion $\mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Z}$.

Choose a smooth atlas $f : Z \to \mathfrak{Z}$ with Z a scheme. Consider the preimage $f^{-1}(U)$ in $Z \times_{\mathfrak{Y}} \mathfrak{X}$. By assumption, since $Z \to \mathfrak{Y}$ is a morphism with source a scheme, there is an open subscheme $W \subset Z$ such that $f^{-1}(U)$ is the preimage of W under the inclusion $Z \times_{\mathfrak{Y}} \mathfrak{X} \to Z$. The image $f(W)$ in $\mathfrak{Z}$ is open, since the morphism f is smooth. By construction, we have that U is the preimage of the open substack $f(W) \subset \mathfrak{Z}$ under the inclusion $\mathfrak{Z} \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Z}$, as desired. □

Proposition 0.4. *Let p be a point of a scheme X. Let $\mathcal{O}_{X,p}$ denote the local ring of X at p. Then, the monomorphism $\operatorname{Spec}(\mathcal{O}_{X,p}) \hookrightarrow X$ is ust.*

Proof. By Proposition 0.3, it suffices to check the required property about subspace topologies for base-changes of morphism $Z \to X$ from a scheme Z. Since the property about the subspace topology can be checked on open covers of Z, X, we can assume without loss of generality that all the schemes X, Z are affine. In this case the monomorphism $\mathrm{Spec}(\mathcal{O}_{X,p}) \to X$ is of the form $\mathrm{Spec}(A_S) \to \mathrm{Spec}(A)$, where A_S is the localization of coordinate ring $A = \mathcal{O}_X$ at a multiplicative set S. If we denote $Z = \mathrm{Spec}(B)$, then the base-changed monomorphism $Z \times_X \mathrm{Spec}(\mathcal{O}_{X,p})$ is of the form $\mathrm{Spec}(B_S) \to \mathrm{Spec}(B)$, where $B_S S$ is the localization at the image of S in B. So it suffices to show that for any ring B and any multiplicative set S, the monomorphism induced by localization $\mathrm{Spec}(B_S) \to \mathrm{Spec}(B)$ induces the subspace Zariski topology on $\mathrm{Spec}(B_S)$. This follows directly from the description of closed subsets in terms of ideals, and the fact that any ideal in B_S is a localization I_S of some ideal $I \subset B$. □

Proposition 0.5. *Let p be a point of a quasiseparated algebraic stack $\mathcal{Y}$. The inclusion $\mathcal{G}_p \hookrightarrow \mathfrak{Y}$ of the residual gerbe at p is ust.*

Proof. Working Zariski locally on $\mathfrak{Y}$, we can assume without loss of generality that $\mathfrak{Y}$ is quasicompact. In this case the topological space of $\mathfrak{Y}$ is sober [**?**, Tag 0DQN] By taking the closure of the point, looking at the corresponding reduced substack, and using that closed immersions are ust, we can furthermore assume that $\mathfrak{Y}$ is irreducible and reduced (integral) and that p is the generic point of $\mathfrak{Y}$. Then we can conclude from Lemma 0.7. □

Let $\mathfrak{Y}$ be an integral quasicompact quasiseparated algebraic stack. Let p denote its generic point, and let $\mathcal{G}_p \to \mathfrak{Y}$ be the residual gerbe of p.

Notation 0.6. *We denote by $\mathcal{G}'_p$ denote the limit (as fibered categories) of all open substacks of $\mathfrak{Y}$. This is just a fibered category satisfying fppf descent and admitting a monomorphism into $\mathfrak{Y}$. By properties of limits, we have a morphism $\mathcal{G}_p \to \mathcal{G}'_p$.*

Lemma 0.7. *Let $\mathfrak{Y}$ be an integral quasicompact quasiseparated algebraic stack. Let p denote its generic point, and let $\mathcal{G}_p \to \mathfrak{Y}$ be the residual gerbe of p.*

(1) The morphism $\mathcal{G}_p \to \mathcal{G}'_p$ described in Notation 0.6 is an isomorphism.

(2) The inclusion $\mathcal{G}_p \to \mathfrak{Y}$ is a ust monomorphism.

Proof. Proof of (1). There is an open (dense) substack $\mathfrak{U} \subset \mathfrak{Y}$ that is a gerbe over a decent algebraic space W [Sta23, Tag 06RC]. Furthermore, there is an open (dense) subspace in W that is a scheme [Sta23, Tag 086U][TAG086U], so by passing to smaller open we can assume without loss of generality that $\mathfrak{Y} \to W$ is a gerbe over an integral scheme W. Let η denote the generic point of W, which is the image of the generic point p. By definition, the inclusion $\eta = \mathrm{Spec}(\mathcal{O}_{W,\eta}) \to W$ is the is the limit over all open subschemes of W. By the construction of the residual gerbe in [Sta23, Tag 006RA], the residual gerbe $\mathcal{G}_p$ is the fiber of η under the morphism $\mathfrak{Y} \to W$. Since limits commute with fiber products, and opens in $\mathfrak{Y}$ correspond to opens in W, we conclude.

Proof of (2). We keep the same notation as in part (1). By Proposition 0.4, we know that the inclusion of the generic point $\eta \to W$. Since ust monomorphisms are preserved by base-change, the base-change $\mathcal{G}_p \to \mathfrak{Y}$ is also a ust monomorphism. □

Remark 0.8. *It follows Lemma 0.7(1) that for an arbitrary quasiseparated stack $\mathfrak{Y}$ and any point p, the residual gerbe $\mathcal{G}_p$ is the limit over all locally closed substacks of $\mathfrak{Y}$ containing the point p.*

Remark 0.9. *Proposition 0.5 is no longer true if we remove the hypothesis that* $\mathfrak{Y}$ *is quasiseparated, even if we assume that the residual gerbe* $\mathcal{G}_p$. *As an example, let* k *be an algebraically closed field of characteristic* 0, *and define* $Y := \mathbb{A}^1_k/\mathbb{Z}$ *to be the algebraic space quotient of* $\mathbb{A}^1_k$ *by the discrete group* $\mathbb{Z}$ *acting by translations (cf. [Sta23, Tag 02Z7]). For any* k*-point* $p : \mathrm{Spec}(k) \to \mathbb{A}^1_k$, *consider its image in the quotient* $p : \mathrm{Spec}(k) \to \mathbb{A}^1_k/\mathbb{Z}$. *The morphism* p *is the inclusion of the residual space of its image. If we take the fiber product with the morphism* $\mathbb{A}^1_k/\mathbb{Z}$ *we obtain the following Cartesian diagram*

$$\begin{array}{ccc} \mathbb{Z} & \hookrightarrow & \mathbb{A}^1_k \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k) & \xrightarrow{p} & \mathbb{A}^1_k/\mathbb{Z} \end{array}$$

Here $\mathbb{Z}$ *has the discrete toplogy, and so the inclusion* $|\mathbb{Z}| \subset |\mathbb{A}^1_k|$ *does not induce the subspace topology on* $\mathbb{Z}$.

References

[Sta23] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`, 2023.