Some flatness results

Andres Fernandez Herrero

Abstract

Here I record some results in commutative algebra/algebraic geometry that are related to the concept of flatness and that I have found useful at some point. This is my laundry list of criteria to try to prove that something is flat.

Proposition 1 (Local criterion for flatness). *See [Sta19, Tag 00ML].*

There are several variations of this. One common one goes as follows.

Proposition 2 (Infinitesimal criterion for flatness). *See [Sta19, Tag 0523].*

Another version of the local criterion is:

Proposition 3 (Slicing criterion for flatness). *See [Sta19, Tag 00ME]*

There are a few of results for checking flatness by passing to fibers.

Proposition 4 (Critere de platitude par fibres). *See [Sta19, Tag 039A].*

Proposition 5 (Flatness of relative complete intersections). *See [Sta19, Tag 00SW].*

There is the miracle flatness criterion, for which you only need to count dimensions.

Proposition 6 (Miracle flatness). *See [Sta19, Tag 00R4].*

There is a criterion for checking flatness over a base in terms of valuation rings, resembling the valuative criteria for properness/separatedness. The reference is [RG71, Corollaire 4.2.10]

Proposition 7 (Valuative criterion for flatness). *Let S be the spectrum of a local reduced ring. Let $p \in S$ denote the special point. Let X be a scheme of finite presentation over S. Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-sheaf of finite type. For any point $x \in X_p$, the following are equivalent*

(1) $\mathcal{F}$ is S-flat at x.

(2) For any spectrum of a valuation ring S' with closed point p' and any local morphism of schemes $(S', p') \longrightarrow (S, p)$, the pullback $\mathcal{F}_{X_{S'}}$ is S'-flat at all points in the special fiber $X_{p'}$ that map to x.

There is another flatness criterion in situations where we expect the morphism to be étale. Recall that a point y in a scheme Y is geometrically unibranch if the strict henselization of $\mathcal{O}_{Y,y}$ has a single minimal prime [Sta19, Tag 06DM] (if Y is reduced, this is the same as $(\mathcal{O}_{Y,y})^{sh}$ being an integral domain).

For the proof of the following, see [Gro67, Thm. 18.10.1]

Proposition 8. *Let $f : X \to Y$ be an unramified morphism of locally Noetherian schemes. Let x be point of X such that*

(1) $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.

(2) $(\mathcal{O}_{Y,f(x)})^{sh}$ is an integral domain.

Then, f is flat at x. In particular, a dominant unramified morphism $f : X \to Y$ between integral Noetherian schemes is automatically étale if Y is geometrically unibranch as in [Sta19, Tag 0BQ2].

The following criterion is inspired in some results on SGA3. The proof uses the valuative criteria for flatness mentioned above.

Proposition 9. *Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes. Suppose that the following conditions are satisfied:*

(1) The morphism $f : X \to Y$ admits a section $\sigma : Y \to X$.

(2) Y is reduced.

(3) The fibers of $f : X \to Y$ are all geometrically integral of the same fixed dimension n.

Then, the morphism $f : X \to Y$ is flat.

For the following, see [Sta19, Tag 052Y]. The result is originally due to Ferrand.

Proposition 10. *Let $X \to Y$ be a finite type morphism of locally Noetherian schemes. Let $f : Z \to Y$ be an integral morphism such that the induced morphism $\mathcal{O}_Y \to f_*(\mathcal{O}_Z)$ is injective. Let $\mathcal{F}$ be a coherent sheaf on X. If the pullback $(f_X)^*\mathcal{F}$ under the base-change $f_X : X \times_Y Z \to Z$ is a Z-flat sheaf on $X \times_Y Z$, then $\mathcal{F}$ is a Y-flat sheaf on X.*

Proposition 11 (Spreading flatness along nilpotent ideals)**.** *See [Sta19, Tag 051J].*

Proposition 12 (Raynaud-Gruson flatification by blowup)**.** *See [Sta19, Tag 0815].*

If you know of other cool criteria for flatness, please let me know!

References

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[RG71] Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Invent. Math.*, 13:1–89, 1971.

[Sta19] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2019.