Local Noether-Lefschetz loci in characteristic 0

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Let $\pi : X \to S$ be a smooth projective morphism of Noetherian $\mathbb{Q}$-schemes (i.e. in characteristic 0). Suppose that the fibers of π are geometrically irreducible.

Under these assumptions, the Picard functor is represented by a group scheme $\mathrm{Pic}_\pi \to S$ locally of finite type over S [BLR90, Thm.3].

Proposition 1. *There is an open subgroup scheme* $\mathrm{Pic}^0_\pi \subset \mathrm{Pic}_\pi$ *such that the structure morphism* $\mathrm{Pic}^0_\pi \to S$ *is proper and has geometrically connected fibers. Furthermore, if S is reduced, then* Pic^0_π *is smooth over S.*

Proof. By our assumptions that the characteristic is 0, each S-fiber of $\mathrm{Pic}_\pi \to S$ is smooth by Cartier's theorem [Sta23, Tag 047N]. Furthermore, by standard deformation theory of line bundles, for any point $s \in S$ with residue field $\kappa(s)$, the dimension of the fiber (Pic_π) is given by the dimension of $H^1(\mathcal{O}_{X_s})$ as a $\kappa(s)$-vector space. The dimension of $H^1(\mathcal{O}_{X_s})$ is known to be locally constant in S [Del68, Thm. 5.5]. Therefore, the existence of Pic^0_π and the smoothness when S follows from [GP11, Exp. VIB, Cor. 4.4]. The properness follows from [BLR90, Thm.3+Thm.4(c)]. We are left to show smoothness. Since the S-fibers of Pic^0_π are connected and the smooth locus of a group scheme is open and closed, it suffices to show that $\mathrm{Pic}^0_\pi \to S$ is smooth at every point of the identity section. By [GP11, Exp.VIB, Prop 1.6], it suffices to show that the conormal bundle $e^*(\Omega^1_{\mathrm{Pic}^0_\pi/S})$ of the unit section $e : S \to \mathrm{Pic}^0_\pi$ is locally free. By deformation theory of line bundles, we have an identification $e^*(\Omega^1_{\mathrm{Pic}^0_\pi/S}) \cong R^1\pi_*(\mathcal{O}_X)$. It follows again from [Del68, Thm. 5.5] that $R^1\pi_*(\mathcal{O}_X)$ is a locally free sheaf, as desired. □

In the following proposition, we describe the connected components of Pic_π étale locally on S.

Proposition 2. *Suppose that S is the spectrum a strictly Henselian local ring. Let* $\mathrm{Pic}_\pi = \sqcup_{i \in I} \mathrm{Pic}^i_\pi$ *denote the decomposition of* Pic_π *into connected components. Then, for each $i \in I$ there exists a closed subscheme $Z_i \subset S$ such that the connected component* Pic^i_π *is a* $\mathrm{Pic}^0_\pi|_{Z_i}$*-torsor over Z_i. In particular,* Pic^i_π *is smooth over Z_i.*

Proof. By 1, there is a proper open group subscheme Pic^0_π that is smooth over S. Choose a π-ample line bundle $\mathcal{O}_X(1)$ on X. The scheme Pic_π breaks as a disjoint union of open and closed projective subschemes, determined by the Hilbert polynomial with respect to $\mathcal{O}_X(1)$. Therefore, each connected component Pic^i_π is of finite type. Since the fibers of $\mathrm{Pic}^0_\pi \to S$ are geometrically connected, the left multiplication action of Pic^0_π on Pic_π preserves Pic^i_π, and so it induces an action morphism $\mathrm{Pic}^0_\pi \times_S \mathrm{Pic}^i_\pi \to \mathrm{Pic}^i_\pi$. The morphism

$$\mathrm{Pic}^0_\pi \times_S \mathrm{Pic}^i_\pi \to \mathrm{Pic}^i_\pi \times_S \mathrm{Pic}^i_\pi, \quad (g, x) \mapsto (g \cdot x, x)$$

is a monomorphism of proper S-schemes, and so it is a closed immersion [Sta23, Tag 04XV]. Therefore, the quotient $\mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi$ is a separated algebraic space of finite type over S (here we may take the quotient because $\mathrm{Pic}^0_\pi \to S$ is a smooth group scheme). Furthermore, $\mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi$ is connected and S-proper, because Pic^i_π is connected and S-proper. For all $s \in S$, the fiber of $\mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi \to S$ over s is a subscheme of the étale group $\mathrm{Pic}_\pi)_s / \mathrm{Pic}^0_\pi)_s$ of connected components of the smooth group scheme $(\mathrm{Pic}^0_\pi)_s$, and so it is unramified. Therefore, $\mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi \to S$ is a proper unramified morphism, and therefore it is finite and $\mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi$ is a scheme by [Sta23, Tag 03XX]. By combining the étale local structure of finite unramified morphisms [Sta23, Tag 04HJ], the connectedness of $\mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi$ and the assumption that S is strictly Henselian, it follows that $Z_i := \mathrm{Pic}^i_\pi / \mathrm{Pic}^0_\pi \to S$ is a closed immersion. By construction Pic^i_π is a $\mathrm{Pic}^0_\pi |_{Z_i}$-torsors over Z_i, as desired. □

We may call the subschemes $Z_i \subset S$ the local Noether-Lefchsetz loci. They are the closed subschemes where the Neron-Severi group of the local family $X \to S$ jumps.

Remark 3. *The number of Z_i might very well be infinite. One may see this, for example, for families of projective $K3$-surfaces.*

Remark 4. *In the case where $X \to S$ is a miniversal projective family of Hyperhkähler manifolds, then the closed subsets Z_i are the preimages of linear subspaces on the period domain. Therefore they are regular. It would be interesting to find a miniversal family of smooth projective varieties where one of the Z_i is singular.*

References

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