October 5, 2022

Dear Gyujin,

I am writing to include a soft proof of a (weak) universal correpresentability result for the de Rham moduli space for big enough characteristics. As you can see, we don't use much about the moduli space in the end.

Fix a genus $g \geq 2$. Let n be a positive integer. For any family of smooth projective geometrically connected curves $C \to S$, we denote by $M\mathrm{Conn}_n(C) \to$ the moduli space of semistable paris (E, ∇), where E is a vector bundle of rank 0 and ∇ is a connection on E.

Proposition 0.1. *There exists a positive integer N depending only on n and g such that the following is satisfied. Let $C \to S$ where S is the spectrum of a DVR. Suppose that for all $s \in S$, we have $char(\kappa(s)) = 0$ or $char(\kappa(s)) \geq N$. Then for any geometric point $\overline{s} \to S$, we have that the natural morphism* $\mathrm{MConn_n}(C \times_S \overline{s}) \to \mathrm{MConn_n}(C) \times_S \overline{s}$ *is an isomorphism.*

Proof. Consider the stack $\mathcal{M}_g$ of polarized smooth projective geometrically connected curves of genus g. Let $\mathcal{C}_{univ} \to \mathcal{M}_g$ denote the universal curve. By smooth descent, there is a representable morphism $\mathrm{MConn_n}(\mathcal{C}_{univ}) \to \mathcal{M}_g$ which is a relative moduli space for connections for the morphism $\mathcal{C}_{univ} \to \mathcal{M}_g$. The stack $\mathcal{M}_g$ is of finite type over $\mathbb{Z}$.

The morphism $\mathrm{MConn_n}(\mathcal{C}_{univ}) \to \mathcal{M}_g$ is representable and of finite type, and $\mathcal{M}_g$ is smooth over $\mathbb{Z}$, and hence reduced. By the last sections in [Sim94], the fibers of $\mathrm{MConn_n}(\mathcal{C}_{univ}) \to \mathcal{M}_g$ over the characteristic 0 points are geometrically irreducible and geometrically normal. By [Sta22, Tag 0559], [Gro66, 12.1.6(iv)], there is a closed substack $\mathcal{Z} \subset \mathcal{M}_g$ with open complement $\mathcal{U}$ such that

(a) $\mathcal{U}$ contains all of the geometric points of characteristic 0 in $\mathcal{M}_g$.

(b) For any geometric point $\overline{u} \to \mathcal{U}$, the base-change $(\mathrm{MConn_n}(\mathcal{C}_{univ}))_{\overline{u}} \to \overline{u}$ is geometrically integral and geometrically normal.

The image of the finite type stack $\mathcal{Z} \to Spec(\mathbb{Z})$ is a constructible subset that does not contain the generic point, and so it must consists of finitely many primes. Let N be a positive integer larger than all of the primes in the image of $\mathcal{Z} \to Spec(\mathbb{Z})$. Then the geometric fibers of the base-change $(\mathrm{MConn_n}(\mathcal{C}_{univ}))_{\mathbb{Z}[\frac{1}{N}]} \to (\mathcal{M}_g)_{\mathbb{Z}[\frac{1}{N}]}$ are geometrically irreducible and integral.

For ease of notation we set $\mathcal{W} = (\mathcal{M}_g)_{\mathbb{Z}[\frac{1}{N}]}$ and denote by $\mathcal{C}$ the base-change of the universal curve to $\mathcal{W}$. We shall show that for any $T \to \mathcal{W}$, where T is either the spectrum of a DVR or the spectrum of a field, the natural morphism $\mathrm{MConn_n}(\mathcal{C} \times_{\mathcal{W}} T) \to \mathrm{MConn_n}(\mathcal{C}) \times_{\mathcal{W}} T$ is an isomorphism. This will conclude the proof of the proposition.

First, we note that under the current assumptions $\mathrm{MConn_n}(\mathcal{C} \times_{\mathcal{W}} T) \to T$ is always flat (this requires a bit of care, you can show that the semistable locus of the stack is integral and it dominates T, and so the same follows for the moduli space). By the fiberwise flatness criterion, we can check that the morphism $\mathrm{MConn_n}(\mathcal{C} \times_{\mathcal{W}} T) \to$

$\mathrm{MConn_n}(\mathcal{C}) \times_{\mathcal{W}} T$ is an isomorphism on geometric fibers. Therefore we thus reduced to the special case when $T = t = Spec(k)$ is the spectrum of a field.

Under our current assumptions, we have that $\mathrm{MConn_n}(\mathcal{C} \times_{\mathcal{W}} t)$ is integral. We also know that the fiber $\mathrm{MConn_n}(\mathcal{C}) \times_{\mathcal{W}} t$ is a normal (irreducible) variety. The morphism $\mathrm{MConn_n}(\mathcal{C} \times_{\mathcal{W}} t) \to \mathrm{MConn_n}(\mathcal{C}) \times_{\mathcal{W}} t$ is finite, and it is an isomorphism on the locus of stable connections. This locus is nonempty by the assumption that $g \geq 2$ (there are stable vector bundles, and so in particular there are stable connections).

We conclude that the morphism $\mathrm{MConn_n}(\mathcal{C} \times_{\mathcal{W}} t) \to \mathrm{MConn_n}(\mathcal{C}) \times_{\mathcal{W}} t$ is a finite birational morphism of (integral) varieties where the target is normal, and so it must be an isomorphism. □

Remark 0.2. *I believe that a similar argument works for S an arbitrary reduced Noetherian scheme. It is probably true for any S and any base-change $T \to S$, but it might not be worth trying to do the more general proof if not needed.*

Remark 0.3. *It might be possible to remove the characteristic assumption and obtain more general results via finer methods (representation theory in positive characteristic, a bit of the geometry of the stack, and etale slices in mixed characteristic...). For example, see the arguments in [VBM08].*

Best regards,

Andres

References

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